Spinors and twistors on loop spaces

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Abstract. This talk reports on joint work of the orator with Mauro Spera on the construction of spinors and twistor spaces over loop spaces, aiming for an analytic approach to elliptic genera via Dirac-type operators on loop spaces extending our work in the flat case to curved riemannian manifolds.

1 Motivation

The "Witten genus" ϕ_W associates a modular form $\phi_W(M)$ to a finite dimensional manifold M. Hypothetically, there should be a "Dirac-Ramond operator" D_K acting on spinor fields over $LM = C^\infty(S^1, M)$, the free loop space of M, such that its S^1-equivariant index equals $\phi_W(M)(q)$, the q-expansion of $\phi_W(M)$, interpreted as a virtual S^1-representation. Killingback and Witten gave a heuristic formula for the operator D_K (see [K] and [Wi] for details):

$$D_K = -i \int_0^1 d\sigma\, \psi^\mu(\sigma) \left[-i \frac{D}{Dx^\mu(\sigma)} + g_{\mu,\nu} \frac{\partial x^\nu}{\partial \sigma} \right] .$$

2 Clifford algebras, spinors and Spin^c-groups in infinite dimensions

Given a real, separable Hilbert space (H, g) one associates to its complexification $H^{\mathbf{C}}$ the hermitian extension $\langle , \rangle$ and the complex bilinear extension $B = g^{\mathbf{C}}$ of g. Furthermore, one has the Clifford algebra $Cl(H, g)$ defined via $[\gamma(u), \gamma(v)]_+ = g(u, v)$, its complexification $\mathbf{C}l(H, g) = Cl(H, g) \otimes \mathbf{C}$, and, given a maximal B-isotropic subspace W of $H^{\mathbf{C}}$, a so-called CAR-algebra. The latter algebra $CAR(W)$ is defined via $[a^*(w_1), a(w_2)]_+ = \langle w_1, w_2 \rangle$, and is isomorphic to $\mathbf{C}l(H, g)$ as a C*-algebra. The algebra $CAR(W)$ is naturally represented on the "spinor space" $S = S_W := \Lambda W$. Elements of the orthogonal group $O(H, g)$ act as automorphisms on $\mathbf{C}l(H, g)$ and are implemented on S if and only if they are in the "restricted orthogonal group" $O_{res}(H, g; W)$. This yields a central S^1-extension $O^{\sim}_{res}(H, g; W)$ which we also denote as $Pin^c(H, g; W)$ (see [SW2] for details).

3 The flat case

Using von Neumann's theory of "incomplete direct products", we construct in [SW1] the Dirac-Ramond operator on

$$L_0\mathbf{R}^d = \left\{ \gamma \in L\mathbf{R}^d \,\middle|\, \int_0^1 \gamma(\sigma) d\sigma = 0 \right\}$$

and show that its S^1-equivariant index exists and equals

$$\left(\prod_{n \geq 1} \frac{1}{1 - q^n} \right)^d .$$

4 Spinors on loop spaces

Recall that a representation $\rho : Spin(2n, \mathbf{R}) \to O(2n, \mathbf{R})$ yields a homomorphism $M_\rho : LSpin(2n, \mathbf{R}) \to O_{res}(H, g; W)$, where $H = L^2(S^1, \mathbf{R}^n)$ and $H^{\mathbf{C}} \supset H_+ = \{L^2\text{−functions extending holomorphically to the unit disc}\}$, and thus there is an induced central S^1-extension $L^{\sim}Spin(2n, \mathbf{R})$ (see [PS]). Given now a principal bundle $Spin(2n, \mathbf{R}) \to P \to M$ and its loopification

$$(*) \;\; LSpin(2n, \mathbf{R}) \to LP \to LM\,,$$

we define a "string structure on P" to be a principal bundle $L^{\sim}Spin(2n, \mathbf{R}) \to L^{\sim}P \to LM$ covering $(*)$. The associated "spinor bundle" is then the associated vector bundle: $\mathcal{S}(LM) = L^{\sim}P \times_{L^{\sim}Spin(2n, \mathbf{R})} S$, where $S = S_{H_+}$ is the spinor space as above. We observe that the obstruction against its existence is exactly a Dixmier-Douady class $DD(P) \in H^3(LM, \mathbf{Z})$.

5 The tangent bundle of loops in a riemannian manifold

For (M, g) a riemannian manifold with Levi-Civita connection ∇, and for γ in LM, with $T_\gamma LM$ identified with $\Gamma_{C^\infty}(S^1, \gamma^* TM)$, there is a linear endomorphism $\nabla_{\dot\gamma}$, the covariant derivative along γ, acting on $T_\gamma LM$. (This operator plays a prominent role in the symplectic geometry of loop spaces, see e.g. [A] and [Wu].) Completing $T_\gamma LM$ with respect to the natural L^2-structure on it yields a Hilbert space $H(\gamma)$ and spectral subspaces $H_\pm(\gamma)$ of $\frac{1}{2\pi i}\nabla_{\dot\gamma}$. The collection $\{H_+(\gamma) \,|\, \gamma \in LM\}$ does not define a "polarisation of LM" (see [Se]) but "the jumps are at most compact operators", more precisely: using explicit trivializations of $(TLM)^{\mathbf{C}}$ over appropriate open subsets $\mathcal{V}_\theta$ of the loop space LM, we show that for γ_1, γ_2 in $\mathcal{V}_\theta$, the difference of the spectral projectors, $p_+(\gamma_1) - p_+(\gamma_2)$, is a Hilbert-Schmidt operator. (See [SW2] for the details.)

6 Twistors on loop spaces

Given a real Hilbert space H as above, one has $G_{res}(H^{\mathbf{C}}, H_+) = \{p \in B(H) \,|\, p^2 = p = p^*$ and p is Hilbert-Schmidt$\}$ and $I_{res}(H, H_+) = \{p \in G_{res} \,|\, Im(p)$ is maximally B-isotropic$\}$, the "restricted grassmannian" resp. the "restricted isotropic grassmannian" (or "restricted twistor space of (H, g)").

We associate to a d-dimensional riemannian manifold M with frame bundle $Q = O(M)$, the fiber bundles $G_{res}(LM) := LQ \times_{LO(d,\mathbf{R})} G_{res}(H^{\mathbf{C}}, H_+)$ and $I_{res}(LM) := LQ \times_{LO(d,\mathbf{R})} I_{res}(H, H_+)$. Furthermore, we have sets (!) with surjective projection onto LM:

$$\widehat{G}_{res}(LM) = \dot{\bigcup}_{\gamma \in LM} G_{res}(H^{\mathbf{C}}(\gamma), H_+(\gamma)) \text{ and}$$

$$\widehat{I}_{res}(LM) = \dot{\bigcup}_{\gamma \in LM} I_{res}(H(\gamma), H_+(\gamma))\,.$$

Using detailed analysis of $(TLM)^{\mathbf{C}}$, we show:

Theorem ([SW2]).

(i) For all riemannian manifolds M, $\widehat{G}_{res}(LM)$ is a smooth, locally trivial fiber bundle and is -as such- isomorphic to $G_{res}(LM)$.

(ii) For all kählerian manifolds M, $\widehat{I}_{res}(LM)$ is a smooth, locally trivial fiber bundle and is -as such- isomorphic to $I_{res}(LM)$.

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