Direct images of secondary classes and renormalization

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I. Chiral anomalies and spectral triples (not new)

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III. Direct images of secondary classes (new)

I. Chiral anomalies and spectral triples

• Let $(\mathscr{C}, H, D)$ be a spectral triple of even degree and finite summability.

$\mathscr{C}$ = Fréchet algebra represented in Hilbert space

$$H = \begin{pmatrix} H_+ \\ H_- \end{pmatrix} , \qquad D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

• Chiral gauge theory: classically

matter fields $\psi \in H_+$, $\overline{\psi} \in (H_-)^*$

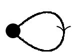
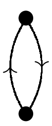
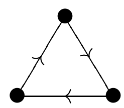
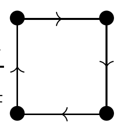
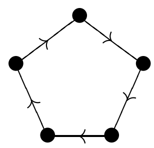
gauge potential $A = \sum_{i,j} c_i^0 [D, c_j^1]$

The classical dynamics of the matter fields evolving in the potential A is governed by the action functional

$$\Gamma^{(0)}(\psi, \overline{\psi}, A_+) = \langle \overline{\psi}, (D_+ + A_+)\psi \rangle$$

Suppose D invertible. The quantum corrections to the classical action (A remains fixed) is the formal power series

$$\Gamma^{(1)}(A_+) = \mathrm{Tr}(D_+^{-1}A_+) - \frac{1}{2}\mathrm{Tr}((D_+^{-1}A_+)^2) + \frac{1}{3}\mathrm{Tr}((D_+^{-1}A_+)^3) \ldots$$

$=$  $-\frac{1}{2}$  $+\frac{1}{3}$  $-\frac{1}{4}$  $+\frac{1}{5}$  $+ \ldots$

Low-order graphs have to be renormalized (heat operator, zeta functions, Dim Reg, etc ...)

• Gauge symmetry: Let $u \in \mathscr{C}\hat{\otimes}_\pi C^\infty(S^1)$ be a smooth loop of invertibles (unitaries) in $\mathscr{C}$.

Gauge transformations of $(\psi_0, \overline{\psi}_0, A_0)$:

$$\psi = u_+^{-1}\psi_0 \ , \qquad \overline{\psi} = \overline{\psi}_0 u_-$$
$$A = u^{-1}[D,u] + u^{-1}A_0 u$$

Fact: the classical action $\Gamma^{(0)}(\psi, \overline{\psi}, A_+)$ is constant over S^1 but not the quantum corrections $\Gamma^{(1)}(A_+)$.

$\Rightarrow$ **anomaly**

The differential of the formal power series $\Gamma^{(1)}(A)$ is a *finite sum*, polynomial in A, linear in $\omega = u^{-1}su$, and **local**. It defines a de Rham cohomology class of the circle

$$s\Gamma^{(1)}(A_+) = \Delta(\omega, A_+) \ \in H^1(S^1)$$

Theorem (P. 2005):

1) Let $\mathscr{A} = \mathscr{C}\hat{\otimes}_\pi C^\infty(S^1)$ and $\mathscr{B} = C^\infty(S^1)$. For any renormalization of $\Gamma^{(1)}(A)$, one has the commutative diagram

$$\begin{array}{ccc}
K_1^{\rm top}(\mathscr{A}) & \longrightarrow & HP_1(\mathscr{A}) \\
{\scriptstyle \mathrm{index}(D)}\downarrow & \searrow^{\Delta} & \downarrow{\scriptstyle \mathrm{ch}(D)} \\
K_1^{\rm top}(\mathscr{B}) & \longrightarrow & HP_1(\mathscr{B}) \\
\| & & \| \\
\mathbb{Z} & & \mathbb{C}
\end{array}$$

2) If $(\mathscr{C}, H, D)$ is *regular* and $\Gamma^{(1)}(A_+)$ is renormalized by zeta-function, the anomaly is a finite sum of residues:

$$\Delta(\omega, A_+) = \operatorname*{Res}_{z=0} \frac{1}{z}\mathrm{Tr}(\gamma\omega|D|^{-2z}) + \sum_{n\geq 1, k\geq 0} (-1)^{n+k} c(k) \times$$

$$\operatorname*{Res}_{z=0} \left(\frac{\Gamma(z+n+k)}{z\Gamma(z)} \mathrm{Tr}((q\omega A^{(k_1)} D A^{(k_2)} \ldots D A^{(k_n)} |D|^{-2(z+n+k)}) \right)$$

$k = (k_1, \ldots, k_n)$ is a multi-index
$q\omega = \frac{1+\gamma}{2}[\omega, D]$
$A^{(k_i)} = k_i$-th commutator of D^2 on A

$c(k)^{-1} = (k_1! \ldots k_n!)(k_1+1)(k_1+k_2+2) \ldots (k_1+\ldots+k_n+n)$

II. Conformal renormalization: simple example

• Smooth étale groupoid $G \ltimes \Sigma$:

$\Sigma = \mathbb{C}$ space of units, G pseudogroup of conformal diffeomorphims $g : \mathrm{Dom}(g) \subset \Sigma \to \mathrm{Ran}(g) \subset \Sigma$

- Either g has finitely many fixed points $z_0 \in \mathrm{Dom}(g)$, with $g'(z_0) \neq 1$,

- Or $g = \mathrm{id}$ on $\mathrm{Dom}(g)$.

Crossed product algebra $\mathscr{C} = C_c^\infty(\Sigma) \rtimes G$ linearly generated by symbols fU_g^*, $f \in C_c^\infty(\mathrm{Dom}(g))$, and product

$$(f_1 U_{g_1}^*)(f_2 U_{g_2}^*) = f_1\,(f_2)^{g_1} U_{g_2 g_1}^*$$

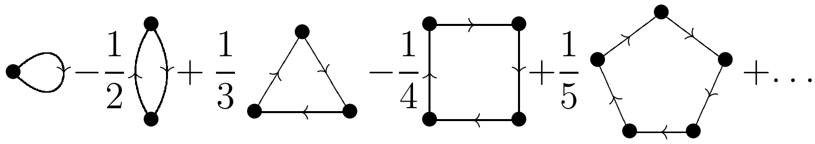
• Dolbeault operator $D_+ = \overline{\partial} = d\overline{z}\partial_{\overline{z}} : C^\infty(\Sigma) \to \Omega^{0,1}(\Sigma)$.

$(\mathscr{C}, D_+)$ is not a spectral triple (no D_-), but D_+ is *conformally invariant.*

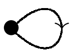
• Conformal field theory: chiral fields $\psi \in C^\infty(\Sigma)$, $\overline{\psi} \in \Omega_c^{1,0}(\Sigma)$ and gauge potential $A_+ = \sum_{i,j} c_i^0 [D_+, c_j^1]$

$$\Gamma^{(0)}(\psi, \overline{\psi}, A_+) = \int_\Sigma \overline{\psi}(D_+ + A_+)\psi$$

$$\text{Propagator:} \quad D_+^{-1}(z,w) = \frac{1}{\pi(z-w)}$$

$\Gamma^{(1)}(A_+) =$ 

Only the first two graphs need renormalization (Operator Product Expansion):

 integrates the kernel $k(z) = \dfrac{1}{g(z) - z}$ for some $g \in G$

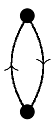
- If g has no fixed point, then $k(z)$ is a smooth function (OK)

- If g has isolated fixed point z_0, with $g'(z_0) \neq 1$. Then k is singular at z_0 with

$$k(z) \sim \frac{1}{(g'(z_0) - 1)} \frac{1}{(z - z_0)} \quad \text{(OK)}$$

- If $g = \text{id}$ on $\text{Dom}(g)$, then $k(z)$ is not defined.

$$k(z) \rightsquigarrow 0$$

 integrates the kernel $k(z, w) = \dfrac{1}{(g(z) - w)(h(w) - z)}$

for some $g, h \in G$.

- If $gh \in G$ has only isolated fixed points OK.

- If $gh = \text{id}$ on $\text{Dom}(gh)$ then around $g(z) \sim w$ the function $k(z, w)$ exhibits a singularity of the kind

$$\frac{1}{(g(z) - w)^2} \rightsquigarrow \partial_w \frac{1}{(g(z) - w)}$$

• The chiral anomaly $\Delta(\omega, A_+) = s\Gamma^{(1)}(A_+)$ may be decomposed as a sum

$$\Delta(\omega, A_+) = \sum_{g \in G} \Delta(\omega, A_+)[g]$$

- If g has isolated fixed points then $(\omega = \sum_g \omega[g]U_g^*)$

$$\Delta(\omega, A_+)[g] = \sum_{z_0} \frac{1}{1 - g'(z_0)}\, \omega(z_0)[g]$$

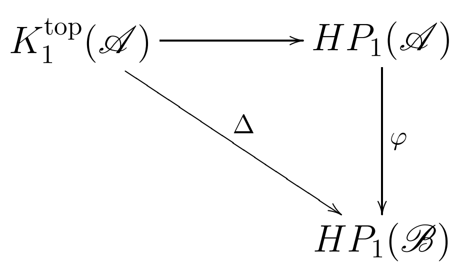
- If $g = \mathrm{id}$ on $\mathrm{Dom}(g)$ then

$$\Delta(\omega, A_+)[g] = \frac{1}{2\pi i} \int_\Sigma (\partial\omega - \frac{1}{2}\,\delta\omega) A_+ \;[g]$$

with $\partial = dz\partial_z$ and $\delta = [\partial, \sigma]$, where σ is the generator of the modular automorphism group of $C_c^\infty(\Sigma) \rtimes G$:

$$\sigma(fU_h^*) = f \ln |h'|^2\, U_h^*$$

Remark: With $\mathscr{A} = C^\infty(\Sigma \times S^1) \rtimes G$ and $\mathscr{B} = C^\infty(S^1)$ there exists a cyclic cocycle $\varphi : HP_1(\mathscr{A}) \to HP_1(\mathscr{B}) = \mathbb{C}$ such that



III. Direct images of secondary classes

All algebras are Fréchet, with topology generated by submultiplicative seminorms: $q(ab) \leq q(a)q(b)$ (needed for Bott periodicity).

Examples: Banach algebras, $C^\infty(M)$, smooth compact operators (matrices with rapid decay over $\mathbb{C}$), etc...

Primary and Secondary classes (Connes, Karoubi): For any such algebra $\mathscr{A}$ there exists a long exact sequence

$$\ldots K_{n+1}^{\text{top}}(\mathscr{A}) \to HC_{n-1}(\mathscr{A}) \xrightarrow{\delta} MK_n(\mathscr{A}) \to K_n^{\text{top}}(\mathscr{A}) \to HC_{n-2}(\mathscr{A}) \ldots$$

For $\mathscr{A} = C^\infty(M)$ the multiplicative K-theory $MK_n(\mathscr{A})$ has close connections with (smooth) Deligne cohomology $\Rightarrow$ geometric invariants.

Quasihomomorphisms (Cuntz): A p-summable quasihomomorphism from $\mathscr{A}$ to $\mathscr{B}$ is a pair of homomorphisms

$$\rho_\pm : \mathscr{A} \rightrightarrows \mathscr{E} \triangleright \mathscr{I} \hat{\otimes}_\pi \mathscr{B}$$

from $\mathscr{A}$ to an algebra $\mathscr{E}$, with $\mathscr{I} \hat{\otimes}_\pi \mathscr{B}$ a (closed) two-sided ideal in $\mathscr{E}$, and $\mathscr{I}$ a p-summable algebra (typically $\mathscr{I} = \mathscr{L}^p(H)$). For any $a \in \mathscr{A}$,

$$\rho_+(a) - \rho_-(a) \in \mathscr{I} \hat{\otimes}_\pi \mathscr{B}$$

Theorem (P. 2006):

1) Let $\rho : \mathscr{A} \rightrightarrows \mathscr{E} \triangleright \mathscr{I}\hat{\otimes}_\pi\mathscr{B}$ be a $(p+1)$-summable quasihomomorphism, with p even integer. Suppose that $K_n^{\text{top}}(\mathscr{I}\hat{\otimes}_\pi\mathscr{B}) \cong K_n^{\text{top}}(\mathscr{B})$ and ρ verifies adequate properties (existence of (pro)-nilpotent extensions etc...). Then one has a transformation of degree $-p$

$$\begin{array}{ccccccc}
\dots K_{n+1}^{\text{top}}(\mathscr{A}) & \longrightarrow & HC_{n-1}(\mathscr{A}) & \longrightarrow & MK_n(\mathscr{A}) & \longrightarrow & K_n^{\text{top}}(\mathscr{A})\dots \\
\downarrow{\scriptstyle \rho_!} & & \downarrow{\scriptstyle \mathrm{ch}(\rho)} & & \downarrow & & {\scriptstyle \rho_!}\downarrow \\
\dots K_{n+1-p}^{\text{top}}(\mathscr{B}) & \longrightarrow & HC_{n-1-p}(\mathscr{B}) & \longrightarrow & MK_{n-p}(\mathscr{B}) & \longrightarrow & K_{n-p}^{\text{top}}(\mathscr{B})\dots
\end{array}$$

2) The bivariant Chern character $\mathrm{ch}(\rho) \in HC^p(\mathscr{A},\mathscr{B})$ is the boundary of a renormalized eta-cochain

$$\mathrm{ch}(\rho) = \partial\eta_R$$

and is given by a *local formula.* The diagonal map

$$\begin{array}{ccc}
K_1^{\text{top}}(\mathscr{A}) & \xrightarrow{\mathrm{ch}_{p+1}} & HC_{p+1}(\mathscr{A}) \\
\downarrow & \overset{\Delta}{\searrow} & \downarrow{\scriptstyle \mathrm{ch}(\rho)} \\
K_1^{\text{top}}(\mathscr{B}) & \longrightarrow & HC_1(\mathscr{B})
\end{array}$$

is and exact generalization of the chiral anomaly (morally, $\Gamma^{(1)} = \eta_R \circ \mathrm{ch}_{p+1}$).