Twisted Poincaré duality for some quadratic Poisson algebras

Letters in Mathematical Physics, 2007

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Metz, 08-11-2007

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Outline

Poisson algebras

Definition

A Poisson algebra is a commutative algebra R endowed with a bilinear bracket $\{.,.\}$ such that

- $(R, \{.,.\})$ is a Lie algebra;
- $\{r,.\}$ is a derivation of R for all $r \in R$.

Example

1. $R = \mathcal{C}^\infty(\mathbb{R}^2)$, coordinates (x,y), $\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}$. Note that $\{x,y\} = 1$, i.e. this bracket comes from the symplectic structure on $\mathbb{R}^2$.
2. [Poisson, 1809] $R = \mathcal{C}^\infty(\mathbb{R}^{2n})$, $\{f,g\} = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial y_i}$.

Algebraic examples

Note that for a finitely generated algebra one only needs to define the bracket on generators.

1. $R = \mathbb{C}[X, Y]$, set $\{X, Y\} = P \in R$. This always defines a Poisson structure. (Poisson cohomology: Roger and Vanhaecke, 2002).
2. $R = \mathbb{C}[X, Y, Z]$, set $\Phi \in R$. Define $\{X, Y\} = \frac{\partial \Phi}{\partial Z}$, $\{Z, X\} = \frac{\partial \Phi}{\partial Y}$, $\{Y, Z\} = \frac{\partial \Phi}{\partial X}$. One may check it defines a Poisson structure, Φ is called the potential of the bracket. (Poisson cohomology: Pichereau, 2006).

Poisson homology and cohomology are important invariants of the Poisson structure.
A whole class of examples comes from the “semi-classical” limit process, which will be described in the sequel.

Poisson modules

Definition

A Poisson module over R is a vector space M endowed with two bilinear maps . and $\{.,.\}_M$ such that

- $(M,.)$ is a (right) module over the commutative algebra R,
- $(M,\{.,.\}_M)$ is a (right) module over the Lie algebra $(R,\{.,.\})$,
- $x.\{a,b\} = \{x,a\}_M.b - \{x.b,a\}_M$ for all $a,b \in R$ and $x \in M$.
- $\{x,ab\}_M = \{x,a\}_M.b + \{x,b\}_M.a$ for all $a,b \in R$ and $x \in M$.

Examples

Let $I \subset R$ be a Poisson ideal. Then I, R/I are Poisson modules.

R Poisson algebra $\longrightarrow$ A associative *noncommutative* algebra with product coming from the Poisson bracket of R.

The semi-classical limit is, roughly speaking, the inverse process, which can be done in the following way.

From now on $k = \mathbb{C}$.
Let A be an algebra such that $\exists \tilde{h} \in Z(A)$ not a zero divisor and such that $\overline{A} = A/\tilde{h}A$ is commutative (i.e. $[A, A] \subseteq \tilde{h}A$).Then set $\{a + \tilde{h}A, b + \tilde{h}A\} = [a, b]/\tilde{h} + \tilde{h}A$.
$(\overline{A}, \{., .\})$ is Poisson, and called the semi-classical limit of A.
A is called a quantification of $\overline{A}$.
For $\lambda \in \mathbb{C}^*$, $A_\lambda := A/(\tilde{h} - \lambda)A$ is called a deformation of $\overline{A}$.

Example

$A = \mathcal{U}(\mathfrak{g})$, $\mathfrak{g} = \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}z$, $[x, y] = z$, z central.Set $\tilde{h} = z$, then $\overline{A} = \mathbb{C}[X, Y]$, $\{X, Y\} = 1$.
$\forall q \in \mathbb{C}^*$, $A_q \simeq A_1(\mathbb{C})$.

Main example

$M = (a_{ij}) \in M_n(\mathbb{Z})$ skew-symmetric.
$A = \mathbb{C}[h^{\pm 1}] < x_i, 1 \leq i \leq n \mid x_i x_j = h^{a_{ij}} x_j x_i >$ is the quantification. (Note that $\tilde{h} = h - 1$)
For $q \in \mathbb{C}^*$, $A_q = A/(h-q)A$ is a quantum affine space (deformation).
The semi-classical limit is $\overline{A} = A/(1-h)A \simeq \mathbb{C}[X_1, \ldots, X_n]$ with bracket as follows.
$\{X_i, X_j\} = \frac{[x_i, x_j]}{h-1} + (h-1)A = \frac{(h^{a_{ij}}-1)x_j x_i}{h-1} + (h-1)A,$

$$\{X_i, X_j\} = a_{ij} X_i X_j.$$

Formally, one could consider that $\{X_i, X_j\} = \frac{[x_i,x_j]}{q-1}\Big|_{q=1}$ in the deformation.

Remark

1. In the rest of the talk, the hypothesis that $a_{ij} \in \mathbb{Z}$ will not play any role.
2. For a monomial $X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}$ we get

$$\{X_1^{\alpha_1} \dots X_n^{\alpha_n}, X_i\} = -\sum_j a_{ij} \alpha_j X^{\alpha+\epsilon_i}.$$

Poisson cohomology.

Now $(R, \{.,.\})$ denotes a Poisson algebra over $\mathbb{C}$.The Poisson cohomology is given by the following complex $\chi^*(R) := \oplus_{k\in\mathbb{N}}\chi^k(R)$, with $\chi^k(R)$ the R-module of all skew-symmetric k-linear derivations of R;

Definition

For any $k \in \mathbb{Z}_{\geq 0}$,
$\chi^k(R) = \{\phi \in \mathrm{Hom}_{\mathbb{C}}(\Lambda^k R, R) \mid \phi(ab, a_2, \ldots, a_k) = a\phi(b, a_2, \ldots, a_k) + b\phi(a, a_2, \ldots, a_k)\}$. It is made an R-module in the obvious way.

Proposition

1. *$\chi^0(R) = R$; $\chi^1(R) = \mathrm{Der}(R, R)$.*
2. *If $R = \mathbb{C}[X_1, \ldots, X_n]$ then $\chi^p(R) = 0$ for all $p > n$.*

Poisson cohomology: the differential

- The Poisson coboundary operator $\delta_k : \chi^k(R) \to \chi^{k+1}(R)$ is defined by

$$\delta_k(P)(f_0,\ldots,f_k) := \sum_{i=0}^{k}(-1)^i \left\{f_i, P(f_0,\ldots,\widehat{f_i},\ldots,f_k)\right\}$$
$$+ \sum_{0\leq i<j\leq k}(-1)^{i+j} P\left(\{f_i,f_j\},f_0,\ldots,\widehat{f_i},\ldots,\widehat{f_j},\ldots,f_k\right)$$

- Poisson cohomology group:

$$HP^k(R) = \mathrm{Ker}\delta_k/\mathrm{Im}\delta_{k-1}.$$

Remark

The Poisson cohomology contains important informations concerning the Poisson structure (Casimir, derivations, deformations...).

Poisson homology.

The Poisson (canonical) homology of R (with value in M) is given by the complex $C_k^{Poiss}(R,M) := M \otimes_R \Omega^k(R)$, with the following

Definition

1. The R-module $\Omega^1(R)$ of Kähler differential forms is generated by the symbols da for all $a \in R$, with relations
   1.1 $d(ab) = adb + bda$;
   1.2 $d(a+b) = da + db$;
   1.3 $d\lambda = 0$ for all $\lambda \in \mathbb{C}$.
2. $\Omega^k(R) = \Lambda_R^k \Omega^1(R)$ is the R-module of Kähler k-differentials.

Proposition

If $R = \mathbb{C}[X_1, \ldots, X_n]$, *then* $\Omega^k(R) \simeq R \otimes_{\mathbb{C}} \Lambda^k V$, *with* $V = \mathbb{C}dX_1 \oplus \cdots \oplus \mathbb{C}dX_n$.

Poisson homology: the differential

- The boundary operator $\partial_k : C_k^{Poiss}(R,M) \to C_{k-1}^{Poiss}(R,M)$ is defined by

$$\partial_k(m \otimes da_1 \wedge \cdots \wedge da_k) =$$
$$= \sum_{i=1}^{k} (-1)^{i+1} \{m, a_i\}_M \otimes da_1 \wedge \cdots \wedge \widehat{da_i} \wedge \cdots \wedge da_k +$$
$$+ \sum_{1 \le i < j \le k} (-1)^{i+j} m \otimes d\{a_i, a_j\} \wedge da_1 \wedge \cdots \wedge \widehat{da_i} \wedge \cdots \wedge \widehat{da_j} \wedge \cdots \wedge da_k$$

- Poisson homology group:

$$HP_k(R,M) = \mathrm{Ker}\partial_k / \mathrm{Im}\partial_{k+1}.$$

Duality

Recall the following notations concerning our main example.

$M = (a_{ij}) \in M_n(\mathbb{Z})$ skew-symmetric.
$R = \mathbb{C}[X_1, \ldots, X_n]$ with bracket

$$\{X_i, X_j\} = a_{ij} X_i X_j.$$

It is the semiclassical limit of the quantum affine space
$U = \mathbb{C} < x_1, \ldots, x_n \mid x_i x_j = q^{a_{ij}} x_j x_i >$, $q \in \mathbb{C}^*$ generic.

Affine plane: first bracket.

Consider the algebra $R_1 = \mathbb{C}[X, Y]$ endowed with the Poisson bracket defined by $\{X, Y\}_1 = 1$.

Proposition

1. *$HP_2(R_1) \simeq \mathbb{C}$, and $HP_k(R_1) = 0$ for all $k \neq 2$;*
2. *$HP^0(R_1) \simeq \mathbb{C}$, and $HP^k(R_1) = 0$ for all $k \geq 1$;*
3. *$HP^k(R_1) \simeq HP_{2-k}(R_1)$ for all $0 \leq k \leq 2$.*

Remark

This kind of duality always holds in the unimodular case.

Affine plane: second bracket.

Now consider the algebra $R_2 = \mathbb{C}[X, Y]$ endowed with the Poisson bracket defined by $\{X, Y\}_2 = XY$.

Proposition

1. *$HP_0(R_2)$ and $HP_1(R_2)$ are infinite-dimensional, and $HP_k(R_2) = 0$ for all $k \geq 2$;*
2. *$HP^0(R_1) \simeq \mathbb{C}$, $HP^1(R_2) \simeq \mathbb{C}^2$, $HP^2(R_2) \simeq \mathbb{C}^2$, and $HP^k(R_2) = 0$ for all $k \geq 3$.*

Main idea. This Poisson structure admits a deformation, namely the quantum affine plane, for which exists a *twisted duality* between the Hochschild homology and cohomology, thanks to a theorem of Van den Bergh.

Duality *à la* Van den Bergh for affine quantum space

- Let $q \in \mathbb{C}^*$ be a non-root of unity.
- Set $U = \mathbb{C}_Q[x_1, \dots, x_n]$, the quantum affine space parametrised by $Q = (q_{ij}) \in M_n(\mathbb{C}^*)$, with $q_{ij} = q^{a_{ij}}$.
- Let σ be the automorphism of U defined by $\sigma(x_i) = p_i x_i$, with $p_i = \prod_j q_{ji}$.
- Let ${}_\sigma U$ denote the U-bimodule that is U as a $\mathbb{C}$-vector space, with product twisted on the left by σ, i.e. $a \cdot u \cdot b = \sigma(a)ub$ for all $a, b \in U$, $u \in {}_\sigma U$.

Theorem (Van den Bergh)

$$HH_*(U, {}_\sigma U) \equiv HH^{n-*}(U, U).$$

The Poisson module M

As a vector space, $M = \mathbb{C}[X_1, \ldots, X_n] = R$, and M is endowed with the following two actions of R:

- the external product "." is just the usual product of R;
- the external bracket $\{.,.\}_M$ is defined by

$$\{m, X_i\}_M := \left. \frac{mx_i - \sigma(x_i)m}{q-1} \right|_{q=1}$$

for all $m \in M$ and $i \in \{1, \ldots, n\}$.In particular, when $m = X_1^{\alpha_1} \ldots X_n^{\alpha_n}$ is a monomial,

$$\{X_1^{\alpha_1} \ldots X_n^{\alpha_n}, X_i\}_M = -\sum_j a_{ij}(\alpha_j - 1)X^{\alpha+\epsilon_i}.$$

A vector space isomorphism.

Thanks to the canonical volume form $dX_1 \wedge \ldots \wedge dX_n$, the set $\chi^k(R)$ of all skew-symmetric k-linear derivations of R is isomorphic as a vector space to $M \otimes_R \Omega^{n-k}(R)$ via an isomorphism denoted by $\dagger$ and defined by:

$$\dagger(P) = \sum_{\sigma \in S_{k,n-k}} \epsilon(\sigma) P(X_{\sigma_1}, \ldots, X_{\sigma_k}) dX_{\sigma_{k+1}} \wedge \cdots \wedge dX_{\sigma_n}$$

for all $P \in \chi^k(R)$. Here we denote by S_n the set of all n-permutations. For all $\sigma \in S_n$, we denote by $\varepsilon(\sigma)$ its sign and we set $\sigma_i := \sigma(i)$. Also $S_{k,n-k}$ denotes the set of those permutations $\sigma \in S_n$ such that $\sigma_1 < \cdots < \sigma_k$ and $\sigma_{k+1} < \cdots < \sigma_n$.

Comparing homology and cohomology.

$$\begin{array}{ccc} \chi^k(R) & \xrightarrow{\quad \dagger \quad} & M \otimes_R \Omega^{n-k}(R) \\ \Big\downarrow \delta_k & \circlearrowleft & \Big\downarrow (-1)^{k+1}\partial_{n-k} \\ \chi^{k+1}(R) & \xrightarrow{\quad \dagger \quad} & M \otimes_R \Omega^{n-k-1}(R) \end{array}$$

This diagram does not commute *a priori*, but

Proposition

For all $P \in \chi^k(R)$, the following equality holds:

$$(\dagger \circ \delta)(P) = (-1)^{k+1}(\partial \circ \dagger)(P).$$

Results

Theorem

For all $k \in \mathbb{N}$*, we have* $HP_k(R, M) \simeq HP^{n-k}(R)$.

Corollary (Monnier)

$$HP^k(R) \simeq \bigoplus_{\substack{|\beta|=n-k \\ \alpha+\beta\in C}} \mathbb{C}X^\alpha dX^\beta,$$

where $C := \{\gamma \in \mathbb{N}^n \mid \gamma_i = 0 \text{ or } \sum_{j=1}^n a_{ij}(\gamma_j - 1) = 0\}$.

Remark

The Poisson cohomology spaces of R are canonically isomorphic to the Hochschild cohomology spaces of its quantisation.

Further readings

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