3-manifolds: coverings, correspondences and noncommutative geometry

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Motivation:

- (A.Connes, M.M.) Spectral correspondences, analogies between RH and QG

- Categorification in knot theory
  (Khovanov homology, Floer homology)

- Geometric correspondences
  (KK theory, motives)

- Algebras of quantum gravity
  (time evolutions, KMS states)

3-manifolds and branched coverings:

Hilden–Montesinos theorem:

Weak: Every compact oriented (smooth) 3-manifold M is a brached cover of S^3 branched along a link $L \subset S^3$

Strong: Every compact oriented (smooth) 3-manifold M is a 3-fold brached cover of S^3 branched along a knot $K \subset S^3$

Universal knots: (Hilden–Lozano–Montesinos)

$\exists K \subset S^3$ such that every M is a branched cover of S^3 branched along K

Non-uniqueness:

$M =$ Poincaré homology sphere: 5-fold cover branched along trefoil, 3-fold cover branched along the $(2,5)$ torus knot, 2-fold cover branched along $(3,5)$ torus knot

$\Rightarrow$ use as correspondences

Geometric correspondences $\mathcal{C}(K, K')$

$$K \subset L \subset S^3 \stackrel{\pi_K}{\longleftarrow} M \stackrel{\pi_{K'}}{\longrightarrow} S^3 \supset L' \supset K'$$

K, K', L, L' embedded *graphs* in S^3
particular case: L, L' links containing K, K'

$\mathcal{C}(K, K) \ni$ trivial unbranched $id : S^3 \to S^3$

Universal knot $K \Rightarrow \mathcal{C}(K, K)$ all 3-manifolds

Virtual correspondences:

$$\mathrm{Hom}_R(K, K') = R[\mathcal{C}(K, K')]$$

$$\phi = \sum_M a_M M$$

finite sums, R commutative ring

Logician's warning:

"Set" of all 3-manifolds not a set!

but ... identifications

Branched covering: representation

$$\sigma : \pi_1(S^3 \smallsetminus L) \to S_n$$

$\Rightarrow$Data $K \subset L \subset S^3 \leftarrow M \to S^3 \supset L' \supset K'$ of correspondences $\mathcal{C}(K, K')$ subset of

$$\bigcup_{n,m,L,L'} \mathrm{Hom}(\pi_1(S^3 \backslash L), S_n) \times \mathrm{Hom}(\pi_1(S^3 \backslash L'), S_m)$$

condition (σ_1, σ_2) define isomorphic (homeo/diffeo) 3-manifolds

Example: $n = m = 3 \Rightarrow L, L'$ covering moves (colored Reidemeister moves) (Piergallini)

Composition of correspondences:

$$\circ : \mathcal{C}(K,K') \times \mathcal{C}(K',K'') \to \mathcal{C}(K,K'')$$

fibered product

$$K \subset L \subset S^3 \stackrel{\pi_K}{\longleftarrow} M \stackrel{\pi_1}{\longrightarrow} S^3 \supset L'_1 \supset K'$$

generic fibers $\#\pi_K^{-1}(x) = n$, $\#\pi_1^{-1}(x) = m$

$$K' \subset L'_2 \subset S^3 \stackrel{\pi_2}{\longleftarrow} \tilde{M} \stackrel{\tilde{\pi}_{K''}}{\longrightarrow} S^3 \supset L'' \supset K''$$

generic fibers $\#\pi_2^{-1}(x) = \tilde{n}$, $\#\pi_{K''}^{-1}(x) = \tilde{m}$

Composition: $\hat{M} = M \circ \tilde{M} := M \times_{K'} \tilde{M}$

$$= \{(x,y) \in M \times \tilde{M} \,|\, \pi_1(x) = \pi_2(y)\}$$

Branched cover

$$L \cup \pi_K \pi_1^{-1}(L_2) \subset S^3 \stackrel{\hat{\pi}_K}{\leftarrow} \hat{M} \stackrel{\hat{\pi}_{K''}}{\rightarrow} S^3 \supset L'' \cup \pi_{K''} \pi_2^{-1}(L_1)$$

generic fibers $\#\hat{\pi}_K^{-1}(x) = n\tilde{n}$, $\#\hat{\pi}_{K''}^{-1}(x) = m\tilde{m}$

Example: M_n n-fold cyclic branched cover

$$M_m \circ M_n = M_{mn}$$

in $\mathcal{C}(O,O)$, $O =$ unknot

Identity element: $\mathbb{U}_K \in \mathcal{C}(K,K)$
trivial $id: S^3 \to S^3$

$$M \times_{K'} S^3 = \{(m,s) \in M \times S^3 \,|\, \pi_2(m) = s\}$$

$$= \bigcup_{s \in S^3} \pi_2^{-1}(s) = M$$

$M \circ \mathbb{U}_{K'} = M$ and $\mathbb{U}_K \circ M = M$

$$K_1 \subset L_1 \subset S^3 \overset{\pi_{11}}{\longleftarrow} M_1 \overset{\pi_{12}}{\longrightarrow} S^3 \supset L_2 \supset K_2$$

$$K_2 \subset L_2' \subset S^3 \overset{\pi_{22}}{\longleftarrow} M_2 \overset{\pi_{23}}{\longrightarrow} S^3 \supset L_3 \supset K_3$$

$$K_3 \subset L_3' \subset S^3 \overset{\pi_{33}}{\longleftarrow} M_3 \overset{\pi_{34}}{\longrightarrow} S^3 \supset L_4 \supset K_4$$

THM: composition is associative

$$\hat{L}_2 \subset S^3 \overset{\hat{\pi}_{232}}{\longleftarrow} \hat{M}_{23} = M_2 \circ M_3 \overset{\hat{\pi}_{234}}{\longrightarrow} S^3 \supset \hat{L}_4,$$

$$J_1 \subset S^3 \overset{\hat{\pi}_{J_1}}{\longleftarrow} \hat{M}_{1(23)} = M_1 \circ \hat{M}_{23} \overset{\hat{\pi}_{J_4}}{\longrightarrow} S^3 \supset J_4,$$

$$\hat{L}_1 \subset S^3 \overset{\hat{\pi}_{121}}{\longleftarrow} \hat{M}_{12} = M_1 \circ M_2 \overset{\hat{\pi}_{123}}{\longrightarrow} S^3 \supset \hat{L}_3$$

$$I_1 \subset S^3 \overset{\hat{\pi}_{I_1}}{\longleftarrow} \hat{M}_{(12)3} = \hat{M}_{12} \circ M_3 \overset{\hat{\pi}_{I_4}}{\longrightarrow} S^3 \supset I_4$$

$$\hat{L}_2 = L_2' \cup \pi_{22}\pi_{23}^{-1}(L_3') \qquad \hat{L}_4 = L_4 \cup \pi_{34}\pi_{33}^{-1}(L_3)$$

$$J_1 = L_1 \cup \pi_{11}\pi_{12}^{-1}(\hat{L}_2) \qquad J_4 = \hat{L}_4 \cup \hat{\pi}_{234}\hat{\pi}_{232}^{-1}(L_2)$$

$$\hat{L}_1 = L_1 \cup \pi_{11}\pi_{12}^{-1}(L_2') \qquad \hat{L}_3 = L_3 \cup \pi_{23}\pi_{22}^{-1}(L_2)$$

$$I_1 = \hat{L}_1 \cup \hat{\pi}_{121}\hat{\pi}_{123}^{-1}(L_3') \qquad I_4 = L_4 \cup \pi_{34}\pi_{33}^{-1}(\hat{L}_3)$$

$$\hat{\pi}_{121}\hat{\pi}_{123}^{-1}(L_3') = \pi_{11}\pi_{12}^{-1}\pi_{22}\pi_{23}^{-1}(L_3') \Rightarrow J_1 = I_1$$

$$\hat{\pi}_{234}\hat{\pi}_{232}^{-1}(L_2) = \pi_{34}\pi_{33}^{-1}\pi_{23}\pi_{22}^{-1}(L_2) \Rightarrow J_4 = I_4$$

Same argument for multiplicities and branching indices

$$\Rightarrow M_1 \circ (M_2 \circ M_3) = (M_1 \circ M_2) \circ M_3$$

Semigroupoid: (small category) $\mathcal{G}$
partially defined associative product

Units: $\gamma \in \mathcal{U}(\mathcal{G})$
$\gamma\alpha = \alpha$ and $\beta\gamma = \beta$ when defined

Source and target $s(\alpha)$, $t(\alpha)$ in $\mathcal{U}(\mathcal{G})$

$$\mathcal{G}_\gamma = \{\alpha \in \mathcal{G} | s(\alpha) = \gamma\}$$

$s(\alpha\beta) = s(\alpha)$, $t(\alpha\beta) = t(\beta)$
compose when $s(\beta) = t(\alpha)$

(not a groupoid: no inverses)

Convolution algebra: $R =$ commutative ring

$R[\mathcal{G}] =$ functions $f : \mathcal{G} \to R$ finite support

$$(f_1 * f_2)(\alpha) = \sum_{\alpha_1, \alpha_2 \in \mathcal{G} : \alpha_1 \alpha_2 = \alpha} f_1(\alpha_1) f_2(\alpha_2)$$

$f = \sum_{\alpha \in \mathcal{G}} a_\alpha \delta_\alpha$, coeff $a_\alpha \in R$, finite sum

Additive category $\mathcal{C}(\mathcal{G})$: $\gamma, \gamma' \in \mathcal{U}(\mathcal{G})$

$$\mathrm{Hom}_R(\gamma, \gamma') \ni \phi = \sum_\alpha a_\alpha \delta_\alpha,$$

$a_\alpha \in R$, $\alpha \in \mathcal{G}$, $s(\alpha) = \gamma$, $t(\alpha) = \gamma'$

$\Rightarrow$ additive envelope $Mat(\mathcal{C}(\mathcal{G}))$
Objects: formal direct sums $\oplus_i \gamma_i$
Morphisms: matrices of $\phi_{ij} \in \mathrm{Hom}_R(\gamma_i, \gamma_j)$

Additive category:
embedded graphs and correspondences

Objects: $Obj(\mathcal{K})$ embedded graphs $K \subset S^3$

Morphisms: $\mathrm{Hom}_{\mathcal{K}}(K, K')$

$$\phi = \sum_M a_M M$$

$$K \subset L \subset S^3 \leftarrow M \to S^3 \supset L' \supset K'$$

(finite sums, $a_M \in \mathbb{Z}$)

If M multi-connected $\Rightarrow$ relation

$$M = M_1 \amalg M_2 \Leftrightarrow M_1 + M_2$$

$Mat(\mathcal{K})$: objects $\oplus_i K_i$

morphisms $\Phi = (\phi_{ij})$

$$\phi_{ij} \in \mathrm{Hom}_{\mathcal{K}}(K_i, K'_j)$$

Convolution algebra of 3-manifolds:

Semigroupoid:

$$\alpha = \left(K \subset L \subset S^3 \stackrel{\pi_K}{\longleftarrow} M \stackrel{\pi_{K'}}{\longrightarrow} S^3 \supset L' \supset K' \right)$$

shorthand: $\alpha = (M, K, K')$

$$s(\alpha) = (\mathbb{U}_K, K, K) \quad t(\alpha) = (\mathbb{U}_{K'}, K', K')$$

$\mathcal{U}(\mathcal{G}) =$ knots

$\mathcal{G}_K =$ 3-manifolds branched cover along K

$\mathbb{C}[\mathcal{G}]$ functions $f : \mathcal{G} \to \mathbb{C}$ finite support

$$(f_1 * f_2)(M) = \sum_{M_1 \circ M_2 = M} f_1(M_1) f_2(M_2)$$

Involution: $\alpha = (M, K, K') \mapsto \alpha^\vee = (M, K', K)$

$$\alpha = \left(K \subset L \subset S^3 \stackrel{\pi_K}{\longleftarrow} M \stackrel{\pi_{K'}}{\longrightarrow} S^3 \supset L' \supset K' \right)$$

$$\alpha^\vee = \left(K' \subset L' \subset S^3 \stackrel{\pi_{K'}}{\longleftarrow} M \stackrel{\pi_K}{\longrightarrow} S^3 \supset L \supset K \right)$$

$$f^\vee(\alpha) = \overline{f(\alpha^\vee)}$$

Time evolution: $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathbb{C}[\mathcal{G}])$

$$\sigma_t(f)(M) := \left(\frac{n}{m}\right)^{it} f(M)$$

for

$$K \subset L \subset S^3 \overset{\pi_K}{\longleftarrow} M \overset{\pi_{K'}}{\longrightarrow} S^3 \supset L' \supset K'$$

generic fibers $\#\pi_K^{-1}(x) = n$, $\#\pi_{K'}^{-1}(x) = m$

$$\sigma_t(f_1 * f_2)(M) = \left(\frac{n}{m}\right)^{it} (f_1 * f_2)(M)$$

$$= \sum_{M_1, M_2 \in \mathcal{G} : M_1 \circ M_2 = M} \left(\frac{n_1}{m_1}\right)^{it} f_1(M_1) \left(\frac{n_2}{m_2}\right)^{it} f_2(M_2)$$

$$= (\sigma_t(f_1) * \sigma_t(f_2))(M)$$

since $n = n_1 n_2$ and $m = m_1 m_2$

$$\sigma_t(f^\vee)(M) = \left(\frac{m}{n}\right)^{it} f^\vee(M) = (\sigma_t(f))^\vee(M)$$

Representations:

Vector space $\mathcal{H}_K$ finitely supported $\xi : \mathcal{G}_K \to \mathbb{C}$
Action:

$$(\rho(f)\xi)(M) = \sum_{M_1 \in \mathcal{G}, M_2 \in \mathcal{G}_K : M_1 \circ M_2 = M} f(M_1)\xi(M_2)$$

Hamiltonian: $\rho(\sigma_t(f)) = e^{itH}\rho(f)e^{-itH}$

$$(H\,\xi)(M) = \log(n)\;\xi(M)$$

where $\pi_K : M \to S^3 \supset L \supset K$ order n

Problem: infinite multiplicities (universal knots)

Equivalence: cobordisms (PL manifolds)

Given two correspondences M_1 and M_2 in $\mathcal{C}(K, K')$

$$K \subset L_1 \subset S^3 \stackrel{\pi_{K,1}}{\longleftarrow} M_1 \stackrel{\pi_{K',1}}{\longrightarrow} S^3 \supset L'_1 \supset K'$$

$$K \subset L_2 \subset S^3 \stackrel{\pi_{K,2}}{\longleftarrow} M_2 \stackrel{\pi_{K',2}}{\longrightarrow} S^3 \supset L'_2 \supset K'.$$

<u>Branched-cover-cobordism:</u> 4-dimensional manifold W with boundary $\partial W = M_1 \cup -M_2$ and branched covering maps

$$S \subset S^3 \times [0,1] \stackrel{q}{\longleftarrow} W \stackrel{q'}{\longrightarrow} S^3 \times [0,1] \supset S'$$

surfaces S, S' in $S^3 \times [0,1]$

$$M_1 = q^{-1}(S^3 \times \{0\}) = q'^{-1}(S^3 \times \{0\})$$

$$M_2 = q^{-1}(S^3 \times \{1\}) = q'^{-1}(S^3 \times \{1\})$$

$$q|_{M_1} = \pi_{K,1}, \quad q'|_{M_1} = \pi_{K',1}$$

$$q|_{M_2} = \pi_{K,2}, \quad q'|_{M_2} = \pi_{K',2}$$

Boundary $\partial S = L_1 \cup -L_2$ and $\partial S' = L'_1 \cup -L'_2$

$$L_1 = S \cap (S^3 \times \{0\}) \quad L_2 = S \cap (S^3 \times \{1\})$$

$$L'_1 = S' \cap (S^3 \times \{0\}) \quad L'_2 = S' \cap (S^3 \times \{1\})$$

Equivalence: $M_1 \sim M_2$ iff
$\exists W$ branched-cover-cobordism $\partial W = M_1 \cup -M_2$

- $M \sim M$: trivial $M \times [0,1]$
- $M_1 \sim M_2 \Rightarrow M_2 \sim M_1$ opposite orientation
- Transitivity: *gluing* of cobordisms

$$W = W_1 \cup_M W_2$$

$\partial W_1 = M_1 \cup -M$, $\partial W_2 = M \cup -M_2$

$$S_1 \subset S^3 \times [0,1] \xleftarrow{q_1} W_1 \xrightarrow{q_1'} S^3 \times [0,1] \supset S_1'$$

$$S_2 \subset S^3 \times [0,1] \xleftarrow{q_2} W_2 \xrightarrow{q_2'} S^3 \times [0,1] \supset S_2'$$

$$S_1 \cup_L S_2 \subset S^3 \times [0,1] \xleftarrow{q_1 \# q_2} W_1 \cup_M W_2 \xrightarrow{q_1' \# q_2'} S^3 \times [0,1] \supset S_1' \cup_{L'} S_2'$$

Compatibility with composition:

$$M_1 \sim M_2 \in \mathcal{C}(K, K'), \quad M_1' \sim M_2' \in \mathcal{C}(K', K'')$$

$$\Rightarrow M_1 \circ M_1' \sim M_2 \circ M_2' \in \mathcal{C}(K, K'')$$

fibered product of cobordisms

$$W_1 \circ W_2 := \{(x,y) \in W_1 \times W_2 | q_1'(x) = q_2(y)\}$$

Branched-cover-cobordism:

$$\partial(W_1 \circ W_2) = \partial W_1 \circ \partial W_2 = (M_1 \circ M_1') \cup -(M_2 \circ M_2')$$

$$\begin{array}{ccccc} & & W_1 \circ W_2 & & \\ & \swarrow^{T_1} & & \searrow^{T_2} & \\ \widehat{S}_1 \subset S^3 \times [0,1] & & & & S^3 \times [0,1] \supset \widehat{S}_2 \end{array}$$

$$\widehat{S}_1 = S_1 \cup q_1(q_1'^{-1}(S_2)) \quad \widehat{S}_2 = S_2' \cup q_2'(q_2^{-1}(S_1'))$$

$$\partial \widehat{S}_1 = I_1 \cup -I_3 \quad \partial \widehat{S}_2 = I_2 \cup -I_4$$

$$T_1^{-1}(S^3 \times \{0\}) = M_1 \circ M_1' = T_2^{-1}(S^3 \times \{0\})$$

$$T_1^{-1}(S^3 \times \{1\}) = M_2 \circ M_2' = T_2^{-1}(S^3 \times \{1\})$$

Additive category $\mathcal{K}_\sim$

Objects: $Obj(\mathcal{K}_\sim)$ = embedded graphs $K \subset S^3$

Morphisms: $Mor(\mathcal{K}_\sim)$ = cobordism classes
$\text{Hom}_\sim(K, K') = \mathbb{Z}[\mathcal{C}_\sim(K, K')] = \mathbb{Z}[\mathcal{C}(K, K')/\sim]$

$$\phi = \sum_{[M]} a_{[M]}[M]$$

Composition

$\text{Hom}_\sim(K, K') \times \text{Hom}_\sim(K', K'') \to \text{Hom}_\sim(K, K'')$

$$[M] = [M_1] \circ [M_2]$$

Time evolution:

$\bar{\mathcal{G}} =$ semigroupoid $\alpha = ([M], K, K')$
cobordism equivalence classes $[M]$

$\mathbb{C}[\bar{\mathcal{G}}] =$ convolution algebra $f : \bar{\mathcal{G}} \to \mathbb{C}$

$$(f_1 * f_2)[M] = \sum_{[M_1]\circ[M_2]=[M]} f_1[M_1] f_2[M_2]$$

$$\sigma_t(f)[M] := \left(\frac{n}{m}\right)^{it} f[M]$$

covering orders well defined on the classes

$$(H\,\xi)[M] = \log(n)\ \xi[M]$$

<u>Finite multiplicities</u>

Classifying spaces for branched coverings
(Neal Brand, 1980)

$$B_k(M) = [M, B_k]$$

set of k-fold branched coverings of M up to branched-cover-cobordism

$$B_k(S^3) = \pi_3(B_k)$$

Rational homotopy type

$$B_k \otimes \mathbb{Q} \simeq \bigvee_{\alpha} K(\mathbb{Q}, 4)$$

α partitions of k

Fibration:

$$K(\pi, j-1) \to \bigvee^{t-1} \Sigma K(\pi, j-1) \to \bigvee^{t} K(\pi, j)$$

$$S^3 \otimes \mathbb{Q} \to \bigvee^{p(k)-1} S^4 \otimes \mathbb{Q} \to B_k \otimes \mathbb{Q}$$

$\Rightarrow \pi_n(B_k) \otimes \mathbb{Q}$

Gives $\pi_n(B_k) \otimes \mathbb{Q} = \mathbb{Q}^D$

$$D = \begin{cases} p(k) & n = 4 \\ Q(\frac{n-1}{3}, p(k) - 1) & n = 1, 4, 10 \\ & \quad \text{mod } 12, \\ & n \neq 1, 4 \\ Q(\frac{n-1}{3}, p(k) - 1) + Q(\frac{n-1}{6}, p(k) - 1) & n \equiv 7 \mod 12 \\ 0 & \text{otherwise} \end{cases}$$

$p(k) = \#$ partitions, $Q(a, b) = \frac{1}{a} \sum_{d|a} \mu(d) b^{a/d}$

$\mu(d) =$ Möbius function

$\Rightarrow \pi_3(B_k)$ finite for all k

Hamiltonian has finite multiplicities $N_n(K)$

$$1 \leq N_n(K) \leq \#\pi_3(B_n)$$

Partition function and Gibbs states

Growth condition on the $\#\pi_3(B_n)$??

$$Z_K(\beta) = \mathrm{Tr}(e^{-\beta H_K}) = \sum_n \exp(-\beta \log(n)) N_n(K)$$

$$\zeta(\beta) \leq Z_K(\beta) \leq \sum_n \#\pi_3(B_n) n^{-\beta}$$

generating function for the $\#\pi_3(B_n)$

If finite summability: extremal KMS_β states $\beta > 1$

$$\varphi_{\beta,K}(f) = Z_K(\beta)^{-1} \mathrm{Tr}(\rho_K(f) e^{-\beta H_K})$$

Weak limits as $\beta \to \infty$

$$\lim_{\beta \to \infty} \varphi_{\beta,K}(f) = f(\mathbb{U}_K)$$

recover embedded graph K

In NCG instead of quotient convolution algebra

2-category: $\mathcal{G}^2$

- Objects: embedded graphs $K \subset S^3$
- 1-morphisms: branched coverings

$$K \subset L \subset S^3 \overset{\pi_K}{\longleftarrow} M \overset{\pi_{K'}}{\longrightarrow} S^3 \supset L' \supset K'$$

- 2-morphisms: branch-cover-cobordisms

$$S \subset S^3 \times [0,1] \overset{q}{\longleftarrow} W \overset{q'}{\longrightarrow} S^3 \times [0,1] \supset S'$$

Horizontal composition: fibered product

$$W_1 \circ W_2 = W_1 \times_{S^3 \times [0,1]} W_2$$

Vertical composition: gluing

$$W_1 \bullet W_2 = W_1 \cup_M W_2$$

Convolution algebra: $\mathbb{C}[\mathcal{G}^2]$

Two product structures:

$$(f_1 \bullet f_2)(W) = \sum_{W=W_1 \bullet W_2} f_1(W_1) f_2(W_2)$$

$$(f_1 \circ f_2)(W) = \sum_{W=W_1 \circ W_2} f_1(W_1) f_2(W_2)$$

One involution for both:

$$f^\dagger(W) = \overline{f(\bar{W}^\vee)}$$

$W \mapsto \bar{W}$ orientation reversal

$$W = W_1 \bullet W_2 \Rightarrow \bar{W} = \bar{W}_2 \bullet \bar{W}_1$$

$$W = W_1 \circ W_2 \Rightarrow \bar{W} = \bar{W}_1 \circ \bar{W}_2$$

$W \mapsto W^\vee$ exchange of covering maps

$$W = W_1 \circ W_2 \Rightarrow W^\vee = W_2^\vee \circ W_1^\vee$$

$$W = W_1 \bullet W_2 \Rightarrow W^\vee = W_1^\vee \bullet W_2^\vee$$

Interesting time evolutions?

Vertical evolution, 1: Hartle-Hawking gravity

Classical (Euclidean) action for gravity

$$S(g) = -\frac{1}{16\pi}\int_W R\,dv - \frac{1}{8\pi}\int_{\partial W} K\,dv$$

$K =$ trace of II fundamental form

Transition amplitude:

$$\langle (M_1, g_1), (M_2, g_2)\rangle = \int e^{iS(g)} D[g]$$

(Lorentzian) sum over cobordisms W

Time evolution:

$$\sigma_t(f)(W, g) = e^{itS(g)} f(W, g)$$

Problem: discontinuity at $W = W_1 \cup_M W_2$
Conditions on metric near ∂W

Formally: KMS_β

$$\varphi_\beta(f) = \frac{\int f(W, g) e^{-\beta S(g)} D[g]}{\int e^{-\beta S(g)} D[g]}$$

Vertical evolution, 2: moduli spaces

Gauge theories on 4-manifolds:
Yang–Mills, Seiberg–Witten

Moduli spaces

$\mathcal{M} = \{$soln's nonlin elliptic equations$\}$/gauge

Linearized theory: deformation complex

$$\Omega^0 \to \Omega^1 \to \Omega^2$$

Ω^0 =infinitesimal gauge

Ω^1 =tangent space to configuration space

Ω^2 =obstructions

$$\mathcal{D} : \Omega^{odd} \to \Omega^{ev}$$

Ind$\mathcal{D}$ = virtual dimension of $\mathcal{M}$

Gluing theorems for moduli spaces:

$$X = X_+ \cup_M X_-$$

$$\mathcal{M}(X) = \mathcal{M}(X_+) \times_{\mathcal{M}(M)} \mathcal{M}(X_-)$$

$$\dim \mathcal{M}(X) = \dim \mathcal{M}(X_+) + \dim \mathcal{M}(X_-) - \dim \mathcal{M}(M)$$

Theory with boundary: (APS, L^2-mod space)

$$W = W_1 \cup_M W_2$$

$\partial W_1 = M_1 \cup -M$, $\partial W_2 = M \cup -M_2$

$$\mathcal{M}(W) = \mathcal{M}(W_1) \times_{\mathcal{M}(M)} \mathcal{M}(W_2)$$

Time evolution: $\delta(W) := \dim \mathcal{M}(W) - \dim \mathcal{M}(M_2)$

$$\sigma_t(f)(W) = \exp(it\delta(W))\; f(W)$$

$$\sigma_t(f_1 \bullet f_2)(W) = \sum_{W = W_1 \bullet W_2} e^{it\delta(W)} f_1(W_1) f_2(W_2)$$

$$= \sum_{W = W_1 \bullet W_2} e^{it\delta(W_1)} f_1(W_1) e^{it\delta(W_2)} f_2(W_2) = \sigma_t(f_1) \bullet \sigma_t(f_2)(W).$$

Can do something similar for horizontal?

Vertical $W = W_1 \cup_M W_2$

$$\mathrm{Ind}\mathcal{D}_W = \mathrm{Ind}\mathcal{D}_{W_1} + \mathrm{Ind}\mathcal{D}_{W_2}$$

Horizontal $X = X_1 \times_Z X_2$
instead of $[\mathcal{D}_X] \in K(X)$ correspondence

$$U \leftarrow X \rightarrow V$$

$$[\mathcal{D}_X] \in KK(U, V)$$

(Connes–Skandalis) geometric correspondences

$$U \leftarrow X = X_1 \times_Z X_2 \rightarrow V$$

Kasparov product = fibered product

$$[\mathcal{D}_X] = [\mathcal{D}_{X_1}] \circ [\mathcal{D}_{X_2}]$$

$$KK(U, Z) \times KK(Z, V) \rightarrow KK(U, V)$$

Index: Connes–Chern character

$$ch_n : K_i(\mathcal{A}) \to HC_{2n+i}(\mathcal{A})$$

$$ch_n : K^i(\mathcal{A}) \to HC^{2n+i}(\mathcal{A})$$

$$\langle ch(e), ch(x) \rangle = \text{index}$$

Cyclic category (Connes)

$$HC^n(\mathcal{A}) = \mathrm{Ext}^n_\Lambda(\mathcal{A}^\natural, \mathbb{C}^\natural)$$

Bivariant character (Connes; Nistor)

$$ch_n : KK^i(\mathcal{A}, \mathcal{B}) \to \mathrm{Ext}^{2n+i}_\Lambda(\mathcal{A}^\natural, \mathcal{B}^\natural)$$

Kasparov product and Yoneda products

$$\mathrm{Tor}^\Lambda_m(\mathbb{C}^\natural, \mathcal{A}^\natural) \otimes \mathrm{Ext}^n_\Lambda(\mathcal{A}^\natural, \mathcal{B}^\natural) \to \mathrm{Tor}^\Lambda_{m-n}(\mathbb{C}^\natural, \mathcal{B})$$

cap products $\Rightarrow$ index

$$\psi = ch(x)\phi, \quad \phi(e \circ x) = \psi(e)$$

Horizontal time evolution

$$S \subset X \leftarrow W \rightarrow X \supset S'$$

$$[\mathcal{D}_W] \in kk(X, X)$$

$$ch_n[\mathcal{D}_W] \in \mathcal{Y} = \oplus_j \mathrm{Ext}^{2n+j}(\mathcal{A}^\natural, \mathcal{A}^\natural)$$

$$ch_n[\mathcal{D}_{W_1}] ch_m[\mathcal{D}_{W_2}] = ch_{n+m}[\mathcal{D}_{W_1 \circ W_2}]$$

$$\chi : \mathrm{Ext}^{2n+j}(\mathcal{A}^\natural, \mathcal{A}^\natural) \rightarrow \mathbb{C}$$

$$\chi ch([\mathcal{D}_{W_1}] \circ [\mathcal{D}_{W_2}]) = \chi ch[\mathcal{D}_{W_1}] \chi ch[\mathcal{D}_{W_2}]$$

$$\sigma_t(f)(W) = |\chi ch[\mathcal{D}_W]|^{it} f(W)$$

$$\sigma_t(f_1 \circ f_2)(W) = (\sigma_t(f_1) \circ \sigma_t(f_2))(W)$$

Note: extends to case of product of manifold W by NC space

Spectral correspondences

$$(\mathcal{A}_1, \mathcal{A}_2, \mathcal{H}, D)$$

$\mathcal{A}_1$, $\mathcal{A}_2$ unital involutive algebras

representations $\rho_i : \mathcal{A}_i \to \mathcal{B}(\mathcal{H})$

$$[\rho_1(a_1), \rho_2(a_2)] = 0, \quad \forall a_1 \in \mathcal{A}_1,\ \forall a_2 \in \mathcal{A}_2$$

D self-adjoint operator, compact resolvent

$$[[D, \rho_1(a_1)], \rho_2(a_2)] = 0, \quad \forall a_1 \in \mathcal{A}_1,\ \forall a_2 \in \mathcal{A}_2$$

$[D, \rho_1(a_1)]$ and $[D, \rho_2(a_2)]$ bounded

Even: $\gamma \in \mathcal{B}(\mathcal{H})$, $\gamma^2 = 1$,

$$D\gamma + \gamma D = 0, \quad [\gamma, \rho_i(a_i)] = 0$$

Note: also allow degenerate D

Geometric and spectral correspondences

$$L_1 \subset S^3 \xleftarrow{\pi_1} M \xrightarrow{\pi_2} S^3 \supset L_2$$

$$S_M := (C^\infty(S^3), C^\infty(S^3), L^2(M,S), \partial\!\!\!/_M)$$

$\mathcal{A}$, $\mathcal{B}$ fin dim unital (noncomm) invol algs
V finite dimensional vector space, $\mathcal{A}$, $\mathcal{B}$ actions
$T \in \mathrm{End}(V) \quad [[T,a],b] = 0, \quad \forall a \in \mathcal{A}, b \in \mathcal{B}$

$$S_F := (\mathcal{A}, \mathcal{B}, V, T)$$

$$S_M \cup S_F = (C^\infty(S^3) \otimes \mathcal{A}, C^\infty(S^3) \otimes \mathcal{B}, \mathcal{H}, D)$$

even S_F: $\mathcal{H} = L^2(M,S) \otimes V$

$$D = T \otimes 1 + \gamma \otimes \partial\!\!\!/_M$$

odd S_F: $\mathcal{H} = L^2(M,S) \otimes V \oplus L^2(M,S) \otimes V$

$$D = \begin{pmatrix} 0 & \delta^* \\ \delta & 0 \end{pmatrix}$$

$\delta = T \otimes 1 + i \otimes \partial\!\!\!/_M$

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Spectral triples with boundary
(Chamseddine-Connes)

$$(\mathcal{A}, \mathcal{H}, D) \Rightarrow (\partial\mathcal{A}, \partial\mathcal{H}, \partial D)$$

Spectral correspondences with boundary

$$(\mathcal{A}_1, \mathcal{A}_2, \mathcal{H}, D) \Rightarrow (\partial\mathcal{A}_1, \partial\mathcal{A}_2, \partial\mathcal{H}, \partial D)$$

spectral correspondence
Note: product of boundaries, corners

Cobordism $S_{F_i} = (A_i, B_i, V_i, T_i)$ and $S_{M_i} \cup S_{F_i}$ cobordant: $S_F = (A, B, V_F, D_F)$ (no boundary)

$$S_W = (C^\infty(S^3 \times I), C^\infty(S^3 \times I), \mathcal{H}_W, \not{D}_W)$$

(with boundary) $\Rightarrow S_W \cup S_F = (\mathcal{A}, \mathcal{B}, \mathcal{H}, \mathcal{D})$ w/boundary

$$(\partial\mathcal{A}, \partial\mathcal{B}, \partial\mathcal{H}, \partial\mathcal{D}) = (\partial S_W) \cup S_F$$

Dirac $\partial\mathcal{D}$: $\not{\partial}_{M_i}$ and T
A–A_i bimodules E_i and B_i–B bimodules F_i w/connections

$$V_i = E_i \otimes_A V \otimes_B F_i$$

T_i from T and connections on E_i, F_i

Compositions:

- Fibered product $\Rightarrow$ Kasparov product

$$(S_{W_1} \cup S_{F_1}) \circ (S_{W_1} \cup S_{F_2})$$

rank E, F $\Rightarrow$ time evolution

- Gluing $W_1 \cup_M W_2$

<u>Problem:</u> gluing theory for spectral triples with boundary: *non-APS boundary conditions!*

$$(S_{W_1} \cup S_{F_1}) \bullet (S_{W_1} \cup S_{F_2}) \,??$$

More questions:

– Other evolutions involving moduli spaces $\mathcal{M}_W$ (non-linear)

- gluing formula
- pullback formula

Donaldson/SW invariants
Gauge theory for embedded surfaces
(Kronheimer-Mrowka)

– Category of complexes over $\mathcal{K}$ or $\mathcal{K}_\sim$
$\Rightarrow$ knot invariants?

– Subshifts of finite type from

$$\sigma : \pi_1(S^3 \setminus L) \to S_m$$

(Silver-Williams) $\Rightarrow$ NC spaces $\mathcal{O}_A$
$(\sigma_1, \sigma_2) \in \mathcal{C}(K, K')$ (covering moves)