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Noncommutative families of instantons

Giovanni Landi

based on
G.L., R. Reina, C. Pagani, W. van Suijlekom: arXiv:0710.0721

A five-parameter family of $SU(2)$ instantons – of topological charge equal to 1 – on a nc S^4_θ . noncommutative parameters

critical points of a gauge functional; satisfy self-duality equations with respect to a Hodge star operator on forms on S^4_θ

obtained used twisted conformal transformations

index theorem give a completeness argument:
the dimension of the “tangent space” to the moduli space is the index of a twisted Dirac operator coupled to the gauge field (and is computed with a nc local index formula)

Projection approach to gauge theory

M. Dubois-Violette $\sim$ 30 years ago

Not difficult to write down projection of the correct rank and of any (integer) topological charge; but they are non instantons, i.e. the corresponding gauge curvature is not self-dual

We do not have yet a full classification for all values of the charge; conting the correct number and a full description of parameters

Toric noncommutative manifolds a.k.a. isospectral deformations

deformations of a classical Riemannian mfld;
satisfy all properties of a nc spin geometry:

Theorem 1. *Let M be a compact Riemannian spin manifold (without boundary) whose isometry group has rank $r \geq 2$. Then M admits natural isospectral deformations to noncommutative geometries M_θ.*

$\theta = (\theta_{ab} = -\theta_{ba}) \qquad \theta_{ab} \in \mathbb{R}$

idea: deform the standard spectral triple describing the Riemannian geometry of M along a torus embedded in the isometry group to get an isospectral triple

$$(C^\infty(M_\theta), H, D, \gamma)$$

Deforming a torus action (M. Rieffel)

M an m dim compact Riemannian spin mfld

an isometric smooth action σ of $\mathbb{T}^n$, $n \geq 2$

decompose $C^\infty(M)$ into spectral subspaces indexed by the dual group $\mathbb{Z}^n = \widehat{\mathbb{T}}^n$: each $r \in \mathbb{Z}^n$ labels a character of $\mathbb{T}^n$

$$e^{2\pi is} \mapsto e^{2\pi ir\cdot s}$$

the r-th spectral subspace for σ on $C^\infty(M)$: functions f_r s.t.

$$\sigma_s(f_r) = e^{2\pi \mathrm{i} r\cdot s} f_r,$$

each $f \in C^\infty(M)$ is the sum of a unique (rapidly convergent) series $f = \sum_{r\in\mathbb{Z}^n} f_r$

$\theta = (\theta_{jk} = -\theta_{kj})$ a real antisymmetric $n \times n$ matrix

the θ-deformation of $C^\infty(M)$: replace the ordinary product by a deformed product. on spectral subspaces is given by

$$f_r \times_\theta g_{r'} := f_r \ \sigma_{\frac{1}{2} r \cdot \theta}(g_{r'}) = e^{\pi \mathrm{i} r \cdot \theta \cdot r'} f_r g_{r'},$$

denote $$C^\infty(M_\theta) := (C^\infty(M), \times_\theta)$$

the action σ of $\mathbb{T}^n$ extends to $C^\infty(M_\theta)$

at the level of the C^*-algebra of continuous functions one has a strict deformation quantization in the direction of the Poisson structure defined by the matrix θ.

the action of $\mathbb{T}^n$ on $C^\infty(M)$ extends to an action on the deRham complex $\Omega^*(M)$ commuting with the exterion derivative d

with the same techniques, one deforms the exterior product, while unchanging d to get a complex

$(\Omega^*(M_\theta), d)$

with $\Omega^0(M_\theta) = C^\infty(M_\theta)$

it is not graded commutative in general

The Hodge operator $*$ on $\Omega^*(M)$ of the Riemannian metric is twisted to an Hodge operator $*_\theta$ on $\Omega^*(M_\theta)$

$\mathcal{H} := L^2(M, \mathcal{S})$ the Hilbert space of spinors; D the Dirac operator of the metric of M; $C^\infty(M)$ act on spinors pointwisely:

$$\Rightarrow \qquad (C^\infty(M), H, D)$$

the canonical spectral triple on M

a double cover $c : \widetilde{\mathbb{T}}^n \to \mathbb{T}^n$ and a representation of $\widetilde{\mathbb{T}}^n$ on $\mathcal{H}$ by unitary operators $U(s)$, s.t

$$U(s) D U(s)^{-1} = D, \qquad U(s) \pi(f) U(s)^{-1} = \pi(\sigma_{c(s)}(f)).$$

$P = (p_1, p_2, \ldots, p_n)$ infinitesimal gen.s of the toric action

$$U(s) = \exp 2\pi \mathrm{i}\, s \cdot P$$

For $T \in \mathcal{B}(\mathcal{H})$ an action $\widetilde{\mathbb{T}}^n \ni s \mapsto \alpha_s(T) := U(s)TU(s)^{-1}$

a spectral decomposition of $T \in \mathcal{B}(\mathcal{H})$: $T = \sum T_r$; $r \in \mathbb{Z}^n$ and T_r is homogeneous of degree r for the action of $\widetilde{\mathbb{T}}^n$:

$$\alpha_s(T_r) = e^{2\pi \mathrm{i} r \cdot s} T_r, \quad \forall \quad s \in \widetilde{\mathbb{T}}^n$$

a twisted representation on $\mathcal{H}$ of the smooth elements of $\mathcal{B}(\mathcal{H})$

$$L_\theta(T) := \sum_r T_r \exp\left\{\pi \mathrm{i}\, r_j \theta_{jk} p_k\right\}$$

in particular we have $L_\theta(C^\infty(M))$;
it is isomorphic, as an algebra, to $C^\infty(M_\theta)$:

$$L_\theta(f \times_\theta g) = L_\theta(f) L_\theta(g)$$

think of L_θ as a *quantization map* from

$$L_\theta : C^\infty(M) \to C^\infty(M_\theta)$$

the datum $(L_\theta(C^\infty(M)), \mathcal{H}, D)$ is a nc spin geometry (also a twisted real structure J and a $\mathbb{Z}_2$-grading γ of $\mathcal{H}$ – for the even case)

all spectral properties are unchanged;

the triples are m^+-summable (of metric dimension m) and there is a noncommutative integral as a Dixmier trace

$$⨍ L_\theta(f) := \mathrm{Tr}_\omega\Big(L_\theta(f)|D|^{-m}\Big)$$

$\mathcal{E}$ a noncommutative vector bundle

a connection on $\mathcal{E}$: a linear map satisfying a Leibniz rule

$$\nabla : \mathcal{E} \mapsto \mathcal{E} \otimes_{C^\infty(M_\theta)} \Omega^1(M_\theta), \qquad \nabla(\eta a) = (\nabla\eta)a + \eta \otimes_{\mathcal{A}} \mathrm{d}a$$

it extends to

$$\nabla : \mathcal{E} \otimes_{C^\infty(M_\theta)} \Omega^p(M_\theta) \mapsto \mathcal{E} \otimes_{C^\infty(M_\theta)} \Omega^{p+1}(M_\theta)$$

the curvature $F = \nabla^2$ is $C^\infty(M_\theta)$-linear; $F \in \mathrm{Hom}_{C^\infty(M_\theta)}(\mathcal{E}, \Omega^2(M_\theta))$

Bianchi identity: $[\nabla, F] = 0$

once the commutator is defined, this is just $\nabla \circ \nabla^2 - \nabla^2 \circ \nabla = 0$.

On $\mathcal{E}$ a natural Hermitian metric $\langle \eta, \xi \rangle \in C^\infty(M_\theta)$,
extended to $\mathcal{E} \otimes_{C^\infty(M_\theta)} \Omega^2(M_\theta))$

a connection ∇ is metric compatible if

$$\langle \nabla\eta, \xi \rangle + (-1)^{|\eta|} \langle \eta, \nabla\xi \rangle = \mathsf{d} \langle \eta, \xi \rangle$$

$C(\mathcal{E})$ is the collection of metric compatible connections on $\mathcal{E}$

for a $\nabla \in C(\mathcal{E})$ with curvature F, the Yang-Mills functional

$$\mathsf{YM}(\nabla) = \int\!\!\!\!\!- *_\theta \mathsf{tr}(F *_\theta F).$$

and tr is the $C^\infty(M_\theta)$-valued trace on $\mathsf{End}_{C^\infty(M_\theta)}(\mathcal{E})$ coming from the Hermitian metric on $\mathcal{E}$

$C(\mathcal{E})$ is an affine space modelled on $\text{Hom}_{C^\infty(M_\theta)}(\mathcal{E}, \Omega^1(M_\theta))$

$$\nabla_t = \nabla + t\alpha$$

the corresponding equations for critical points of $\text{YM}(\nabla)$:

$$[\nabla, *_\theta F] = 0$$

(antiself)self-dual configrtns are solutions from Bianchi identity

$$*_\theta F = \pm F$$

instantons

The gauge group for $\mathcal{E}$ is the group $\mathcal{G} = \mathcal{U}(\text{End}_{C^\infty(M_\theta)}(\mathcal{E}))$ acts on $C(\mathcal{E})$ by conjugation leaving the functional YM invariant

minima are left invariant under the action of $\mathcal{G}$; the $\mathcal{G}$-orbits in the collection of minima is the 'moduli space' $\mathcal{M}$ of minima

An instanton

a complex ‘rank 2’ vector bundle over S^4 with a self-dual or anti-self-dual connection

A noncommutative instanton

a complex ‘rank 2’ vector bundle over S^4_θ with a self-dual or anti-self-dual connection

In both cases

minima of a Yang-Mills functional;

they have ‘topological numbers’

The example of $SU(2)$ instantons

Classically:

an action of $SU(2)$ on S^7 with quotient S^4

equivariance with a right action of $Spin(5)$

for any finite dim. repr. of $SU(2)$ there is a $Spin(5)$-equivariant vector bundle on S^4

the fundamental representation on $\mathbb{C}^2$ gives the classical 'intanton bundle' where instantons connection live

A nc toric $SU(2)$-fibration

for a θ' determined by θ, an action of ordinary SU(2) on $C^\infty(S^7_{\theta'})$ so that the fixed-point subalgebra is $C^\infty(S^4_\theta)$

now equivariance is with respect to a coaction of $SO_\theta(5)$ or dually, an action of $\mathrm{U}_\theta(so(5))$

representations of $SU(2)$ give projective modules over $C^\infty(S^4_\theta)$

the fundamental representation give the 'noncommutative instanton bundle' $\mathcal{E}$ (say)

GL, W. van Suijlekom
CMP 260 (2005) 203-225 ; 271 (2007) 591–634

Interesting examples with $SU_q(2)$ as ‘structure quantum group’

$SU_q(2)$ co-acts on a quantum sphere S_q^7

coming from the symplectic groups $Sp_q(2)$

the co-fixed-point subalgebra is a quantum sphere S_q^4

here $q \in \mathbb{R}$

GL, C. Pagani, C. Reina
CMP 263 (2006) 65-88

Fix ρ a finite-dim rep of SU(2) on the v.s. W

the corresponding equivariant maps make up $C^\infty(S^4_\theta)$-bimodules

$$\mathcal{E} := C^\infty(S^7_{\theta'}) \boxtimes_\rho W := \Big\{ \eta \in C^\infty(S^7_{\theta'}) \otimes W \\ : (\alpha_w \otimes \mathrm{id})(\eta) = (\mathrm{id} \otimes \rho(w)^{-1})(\eta),\ \forall w \in \mathrm{SU}(2) \Big\},$$

The instanton-bundle: $W = \mathbb{C}^2$ and ρ is the defining rep of SU(2)

$$C^\infty(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^2 \simeq p\Big(C^\infty(S^4_\theta)\Big)^4$$

$$p = \frac{1}{2} \begin{pmatrix} 1 + z_0 & 0 & z_1 & -\bar{\mu} z_2^* \\ 0 & 1 + z_0 & z_2 & \mu z_1^* \\ z_1^* & z_2^* & 1 - z_0 & 0 \\ -\mu z_2 & \bar{\mu} z_1 & 0 & 1 - z_0 \end{pmatrix} \tag{1}$$

The polynomial functions $\mathcal{A}(S^4_\theta)$ are generated by $z_0 = z_0^*$, z_j, z_j^*, $j = 1, 2$, with relations

$$z_\mu z_\nu = \lambda_{\mu\nu} z_\nu z_\mu, \quad z_\mu z_\nu^* = \lambda_{\nu\mu} z_\nu^* z_\mu$$

and $\sum_\mu z_\mu^* z_\mu = 1$; $\theta \in \mathbb{R}$ and the deformation parameters are

$$\lambda_{12} = \bar{\lambda}_{21} =: \lambda = e^{2\pi i\theta},$$
$$\lambda_{j0} = \lambda_{0j} = 1, \quad j = 1, 2, \qquad \mu = \sqrt{\lambda}.$$

The polynomial functions on the sphere $\mathcal{A}(S^7_{\theta'})$ is generated by elements ψ_a, ψ_a^*, $a = 1, \ldots, 4$, with relations

$$\psi_a \psi_b = \lambda'_{ab} \psi_b \psi_a, \quad \psi_a \psi_b^* = \lambda'_{ba} \psi_b^* \psi_a$$

and $\sum_a \psi_a^* \psi_a = 1$.

The parameters are related by

$$\lambda'_{ab} = \begin{pmatrix} 1 & 1 & \overline{\mu} & \mu \\ 1 & 1 & \mu & \overline{\mu} \\ \mu & \overline{\mu} & 1 & 1 \\ \overline{\mu} & \mu & 1 & 1 \end{pmatrix}, \quad \mu = e^{2\pi \mathrm{i}\theta'}, \quad \theta' = \frac{\theta}{2}$$

We can 'deconstruct' $p = \Psi\Psi^\dagger$, with

$$\Psi^\dagger = \begin{pmatrix} \psi_1^* & \psi_2^* & \psi_3^* & \psi_4^* \\ -\psi_2 & \psi_1 & -\psi_4 & \psi_3 \end{pmatrix},$$

which is such that $\Psi^\dagger\Psi = \mathbb{I}_2$. We get

$$z_\alpha = \sum_{ab} \psi_a^* (\gamma_\alpha)_{ab} \psi_b, \qquad z_\alpha^* = \sum_{ab} \psi_a^* (\gamma_\alpha^*)_{ab} \psi_b,$$

with γ_α twisted 4×4 Dirac matrices,

$$\gamma_1 = 2\begin{pmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \\ \begin{smallmatrix} \mu & 0 \\ 0 & 0 \end{smallmatrix} & 0 \end{pmatrix}, \quad \gamma_2 = 2\begin{pmatrix} 0 & \begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & \overline{\mu} \\ 0 & 0 \end{smallmatrix} & 0 \end{pmatrix}.$$

$$\gamma_0 = -\frac{1}{4}[\gamma_1, \gamma_1^*][\gamma_2, \gamma_2^*] = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Twisted Clifford algebra relations,

$$\gamma_j \gamma_k + \lambda_{jk} \gamma_k \gamma_j = 0, \qquad \gamma_j \gamma_k^* + \lambda_{kj} \gamma_k^* \gamma_j = 4\delta_{jk}$$

Compatible toric actions on S^4_θ and $S^7_{\theta'}$

the $\mathbb{T}^2$ action on $\mathcal{A}(S^4_\theta)$:

$$\sigma_s(z_0, z_1, z_2) = (z_0, e^{2\pi i s_1} z_1, e^{2\pi i s_2} z_2), \quad s \in \mathbb{T}^2;$$

a lift to a double cover action on $\mathcal{A}(S^7_{\theta'})$:

$$\widetilde{\mathbb{T}}^2 \to \mathbb{T}^2; \quad (s_1, s_2) \mapsto (s_1 + s_2, -s_1 + s_2)$$

the $\widetilde{\mathbb{T}}^2$ action on $\mathcal{A}(S^7_{\theta'})$

$$\widetilde{\sigma} : (\psi_1, \psi_2, \psi_3, \psi_4) \mapsto \left(e^{2\pi i s_1}\ \psi_1,\ e^{-2\pi i s_1}\ \psi_2,\ e^{-2\pi i s_2}\ \psi_3,\ e^{2\pi i s_2}\ \psi_4\right).$$

The double cover comes from a spin cover $\mathrm{Spin}_\theta(5)$ of $\mathrm{SO}_\theta(5)$ deforming the action of $\mathrm{Spin}(5)$ on S^7 as a double cover of the action of $\mathrm{SO}(5)$ on S^4.

For $e \in K_0(\mathcal{A})$, the Chern character

$$\mathrm{ch}_*(e) = \sum_{k \geq 0} \mathrm{ch}_k(e)$$

is a even cycle in $CC_*(\mathcal{A})$, $(b+B)\mathrm{ch}_*(e) = 0$

b is the Hochschild operator; B is the Connes operator

For $k = 0$: $\quad \mathrm{ch}_0(e) := \mathrm{tr}(e)$,

for $k \neq 0$: $\quad \mathrm{ch}_k(e) := (-1)^k \frac{(2k)!}{k!} \sum (e_{i_0 i_1} - \frac{1}{2}\delta_{i_0 i_1}) \otimes e_{i_1 i_2} \otimes \cdots \otimes e_{i_{2k} i_0}$

represented by bounded operators on the Hilbert space $\mathcal{H}$:

$$\pi_D(\mathrm{ch}_k(e)) = \lambda_k \sum (\pi(e_{i_0 i_1}) - \frac{1}{2}\delta_{i_0 i_1})[D, \pi(e_{i_1 i_2})] \cdots [D, \pi(e_{i_{2k} i_0})].$$

for the instanton projection (1) on S^4_θ

$$\begin{aligned} \mathrm{ch}_0(p) &= 2 \quad \text{(this is the rank)} \\ \mathrm{ch}_1(p) &= 0 \qquad \Rightarrow b(\mathrm{ch}_2(p)) = 0 \end{aligned}$$

the Hochschild cycle $\mathrm{ch}_2(p)$ is 'the volume form on S^4_θ'

with the geometry $(C^\infty(S^4_\theta), D, \mathcal{H}, \gamma_0)$, a quartic equation in D

$$\pi_D(\mathrm{ch}_2(p) = 3\gamma_0,$$

The topological content of $[p]$ is the index $\mathrm{Top}([p]) = \text{index } (D_p)$ of the twisted Dirac operator $D_p = p(D \otimes \mathbb{I}_4)p$

$$\mathrm{ch}_1(p) = 0 \quad \Rightarrow \quad \text{index } (D_p) = ⨍ \gamma_0 \pi_D(\mathrm{ch}_2(p))$$

$$\text{and} \qquad \mathrm{Top}([p]) = 3 ⨍ 1 = 3\mathrm{Tr}_\omega |D|^{-4} = 3\frac{1}{3} = 1$$

The canonical connection $\nabla_0 = p \circ \mathsf{d}$ on equivariant maps $\mathcal{E}$ is

$$(\nabla_0 f)_i = \mathsf{d} f_i + \omega_{ij} \times_\theta f_j$$

with the gauge potential

$$\omega = \Psi^\dagger \mathsf{d} \Psi$$

a 2×2-matrix with entries in $\Omega^1(S^7_{\theta'})$ and $\overline{\omega_{ij}} = \omega_{ji}$ and $\sum_i \omega_{ii} = 0$

ω_{ij} are $\mathbb{T}^2$-invariant and hence central (as one forms) in $\Omega(S^7_{\theta'})$

the curvature $F_0 = \nabla_0^2 = \mathsf{d}\omega + \omega^2$ is in $\mathrm{End}(\mathcal{E}) \otimes_{C^\infty(S^4_\theta)} \Omega^2(S^4_\theta)$ and is self-dual

$$*_\theta F_0 = F_0,$$

remember the computation of its topological charge:

$$\mathrm{Top}([p]) = \mathrm{index}\ (D_p) = 1$$

A noncommutative family of instantons

Classically, charge 1 instantons are generated from the basic one by the action of the conformal group $\mathrm{SL}(2,\mathbb{H}) \simeq \mathrm{SO}(5,1)$ of S^4.

The subgroup $\mathrm{Sp}(2) \subset \mathrm{SL}(2,\mathbb{H})$ leave invariant the basic one; hence to get new instantons one needs to quotient $\mathrm{SL}(2,\mathbb{H})$ by the spin group $\mathrm{Sp}(2) \simeq \mathrm{Spin}(5)$.

The moduli space of $\mathrm{SU}(2)$ instantons on S^4 modulo g.t. is the five-dimensional quotient manifold $\mathrm{SL}(2,\mathbb{H})/\mathrm{Sp}(2)$.

In a parallel attempt, to generate instantons on $\mathcal{A}(S^4_\theta)$, we use a deformed conformal group $\mathrm{SL}_\theta(2,\mathbb{H})$ and its deformed spin subgroup $\mathrm{Sp}_\theta(2)$ (cf. Connes–Dubois-Violette).

These are dual to $\mathrm{U}_\theta(so(5,1))$ and $\mathrm{U}_\theta(so(5))$

We obtain a noncommutative family of instantons:

instantons parametrized by the quotient space of the quantum conformal group $\mathrm{SL}_\theta(2,\mathbb{H})$ by the quantum spin group $\mathrm{Sp}_\theta(2)$

The torus $\mathbb{T}^2$ is embedded in $\mathsf{SL}(2,\mathbb{H})$ via the map ρ

$$\rho(t) = \begin{pmatrix} e^{2\pi \mathrm{i} t_1} & \\ & e^{-2\pi \mathrm{i} t_2} \end{pmatrix}, \qquad t = (t_1, t_2) \in \mathbb{T}^2,$$

$\mathbb{T}^2 \times \mathbb{T}^2$ acts on $\mathsf{SL}(2,\mathbb{H})$ by conjugation:

$$(t, g, s) \in \mathbb{T}^2 \times \mathsf{SL}(2,\mathbb{H}) \to \rho(t) \cdot g \cdot \rho(s)^{-1} \in \mathsf{SL}(2,\mathbb{H}).$$

Any $f \in \mathcal{A}(\mathsf{SL}(2,\mathbb{H}))$ is expanded in a series $f = \sum_r f_r$ of homogeneous elements for this action

A deformed product $\times_\theta$ on spectral subspaces; extended linearly

The resulting algebra $\mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H})$, with the classical coproduct Δ, counit ϵ and antipode S becomes a Hopf algebra.

The defing matrix of $\mathsf{SL}_\theta(2,\mathbb{H})$

$$A_\theta = \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ -a_2^* & a_1^* & -b_2^* & b_1^* \\ c_1 & c_2 & d_1 & d_2 \\ -\bar{c}_2 & \bar{c}_1 & -d_2^* & d_1^* \end{pmatrix} . \tag{2}$$

For the torus action, entries A_{ij} are of bidegree $\Lambda_i - \Lambda_j$, where

$$\Lambda = (\Lambda_i) = ((1,0),(-1,0),(0,1),(0,-1)) .$$

The general strategy exemplified by the deformed product would then give the deformed product and the, in turn, the commutation relations defining the deformed algebra $\mathcal{A}(M_\theta(2,\mathbb{H}))$.

One can get them from the coaction on the algebra $\mathcal{A}(\mathbb{C}^4_\theta)$

One finds commutation relations:

$$A_{ij}A_{kl} = \eta_{ki}\eta_{jl}A_{kl}A_{ij}$$

Proposition 2. *Let* I *denote the two-sided* $*$*-ideal in* $\mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H}))$ *generated by the elements* $\sum_k (A^*)_{ik}A_{kj} - \delta_{ij}$ *for* $i,j = 1,\ldots,4$. *Then* $I \subset \mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H}))$ *is a Hopf ideal.*

The quotient $\mathcal{A}(\mathsf{Sp}_\theta(2)) := \mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H}))/I$ *is a Hopf algebra with the induced Hopf algebra structure.*

The 'defining matrix' A_θ of $\mathcal{A}(\mathsf{Sp}_\theta(2))$ has the form (2) with the additional condition that $A_\theta^* A_\theta = 1$; also $A_\theta A_\theta^* = 1$. These conditions are equivalent to the statement that $S(A_\theta) = A_\theta^*$.

The action on the principal bundle

The 'matrix of equivariant maps' $\Psi = (\Psi_{ia})$, with $i, j = 1, \ldots, 4$ and $a = 1, 2$, consists of two deformed quaternions.

The natural left coaction Δ_L of $\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H}))$

$$\Delta_L : \Psi_{ia} \mapsto A_{ij} \otimes \Psi_{ja} =: \widetilde{\Psi}_{ia}$$

does not preserve the sphere relation:

$$\Delta_L(\sum_a z_a^* z_a) \neq 1 \otimes 1.$$

Denote by $\mathcal{A}(\widetilde{S}^7_\theta)$ the image of $\mathcal{A}(S^7_\theta)$ under the left coaction of $\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H}))$: it is a subalgebra of $\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(S^7_\theta)$.

Think of $\mathcal{A}(\widetilde{S}^7_\theta)$ as a θ-deformation of a whole family of 'inflated' 7-dimensional sphere: the central element

$$\rho^2 := \Delta_L(\sum_a z_a^* z_a)$$

in $\mathcal{A}(\widetilde{S}^7_\theta)$ parametrizes a family of noncommutative 7-spheres $\widetilde{S}^7_\theta$;

by evaluating ρ^2 as a real number $r^2 \in \mathbb{R}$, we obtain a deformation $\mathcal{A}(S^7_{\theta,r})$ of the algebra of polynomials on a 7-sphere of radius r.

As expected, the coaction of the quantum subgroup $\mathcal{A}(\mathrm{Sp}_\theta(2))$ does not 'inflate the spheres', i.e. $\rho^2 = 1 \otimes 1$ in this case.

Both $\mathcal{A}(S^7_\theta)$ and $\mathcal{A}(S^4_\theta)$ are $\mathcal{A}(\mathrm{SL}_\theta(2, \mathbb{H}))$ comodule algebras.

Next, a right action of SU(2) on $\mathcal{A}(\widetilde{S}^7_\theta)$ in such a way that the corresponding algebra of invariants describes a family of noncommutative 4-spheres. It is natural to require that this action commutes with the above left coaction of $\mathcal{A}(\mathrm{SL}_\theta(2,\mathbb{H}))$ on $\mathcal{A}(S^7_\theta)$.

The coaction of $\mathcal{A}(\mathrm{SL}_\theta(2,\mathbb{H}))$ is extended to the forms $\Omega(S^4_\theta)$ by requiring that it commute with d:

$$\Delta_L(\mathrm{d}\omega) = (\mathrm{id} \otimes \mathrm{d})\Delta_L(\omega).$$

Proposition 3. *With $*_\theta$ the isospectral Hodge operator on S^4_θ, the algebra $\mathcal{A}(\mathrm{SL}_\theta(2,\mathbb{H}))$ coacts by conformal transformations:*

$$\Delta_L(*_\theta\omega) = (\mathrm{id} \otimes *_\theta)\Delta_L(\omega)\ , \qquad \forall\, \omega \in \Omega(S^4_\theta)\ .$$

A noncommutative family of instantons on S^4_θ

Out of the matrix valued function Ψ gets a projection $p = \Psi^*\Psi$ whose Grassmannian connection $\nabla = p\,\mathrm{d}$ is self-dual:

$$*_\theta \nabla^2 = \nabla^2$$

The $\mathcal{A}(S^4_\theta)$-module $\mathcal{E} = p[\mathcal{A}(S^4_\theta)]^4$ of noncommutative sections is isomorphic to the $\mathcal{A}(S^4_\theta)$-module of equivariant maps for the defining representation ρ of SU(2) on $\mathbb{C}^2$:

$$\mathcal{E} \simeq \mathcal{A}(S^7_\theta) \boxtimes_\rho \mathbb{C}^2$$
$$:= \Big\{ f \in \mathcal{A}(S^7_\theta) \otimes \mathbb{C}^2 : (\mathrm{id} \otimes \rho(g)^{-1})(f) = (\alpha_g \otimes \mathrm{id})(f) \Big\},$$

Connection $\nabla = p\,\mathsf{d} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}(S^4)} \Omega(S^4_\theta)$ on equivariant maps:

$$\nabla(f_a) = \mathsf{d} f_a + \sum_b \omega_{ab} f_b, \qquad a,b = 1,2.$$

$$\omega_{ab} = \frac{1}{2} \sum_k \Big((\Psi^*)_{ak} \mathsf{d} \Psi_{kb} - \mathsf{d}(\Psi^*)_{ak} \Psi_{kb} \Big).$$

$\omega_{ab} = -\overline{\omega}_{ba}$ and $\sum_a \omega_{aa} = 0$ so that ω is in $\Omega^1(S^7_\theta) \otimes su(2)$.

Out of the coaction of the quantum group $\mathsf{SL}_\theta(2,\mathbb{H})$, we shall get a family of such connections

A family of projections

With A_θ the defining matrix of $\mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H}))$,

$$\widetilde{\Psi} := \Delta_L(\Psi) = A_\theta \,\dot{\otimes}\, \Psi \tag{3}$$

such that

$$(\widetilde{\Psi})^*\widetilde{\Psi} = \Delta_L(\Psi^*\Psi^*) = \Delta_L(\sum_a z_a^* z_a \mathbb{I}_2) = \rho^2 \mathbb{I}_2.$$

With the extra central element ρ^{-2}, a natural $*$-s.adj idempotent:

$$P := \widetilde{\Psi}\rho^{-2}(\widetilde{\Psi})^*$$

Explicitly,

$$P \;=\; \frac{1}{2}\rho^{-2}\begin{pmatrix} \rho^2+\widetilde{x} & 0 & \widetilde{\alpha} & \widetilde{\beta} \\ 0 & \rho^2+\widetilde{x} & -\,\mu\widetilde{\beta}^* & \overline{\mu}\widetilde{\alpha}^* \\ \widetilde{\alpha}^* & -\,\overline{\mu}\widetilde{\beta} & \rho^2-\widetilde{x} & 0 \\ \widetilde{\beta}^* & \mu\widetilde{\alpha} & 0 & \rho^2-\widetilde{x} \end{pmatrix}.$$

Entries are in $\mathcal{A}(\widetilde{S}^4_\theta)$ that is $\mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H})) \otimes \mathcal{A}(S^4_\theta)$; they generate a family of noncommutative 4-spheres

Interpret P as a *noncommutative family of projections*, that is noncommutative vector bundles, parametrized by the noncommutative space $\mathsf{SL}_\theta(2,\mathbb{H})$

Classicaly, at $\theta = 0$, evaluation maps $ev_x : \mathcal{A}(\mathsf{SL}(2,\mathbb{H})) \to \mathbb{C}$ and for each point x in $\mathsf{SL}(2,\mathbb{H})$, $(ev_x \otimes \mathrm{id})P$ is a projection in $\mathsf{Mat}_4(\mathcal{A}(S^4))$, that is a bundle over S^4

The projection P yields a noncommutative family of instantons.

Extend to $\mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H})) \otimes \mathcal{A}(S^4_\theta)$ the differential as $(\mathsf{id} \otimes \mathsf{d})$, and similarly, the Hodge star operator of $\mathcal{A}(S_\theta)$ as $(\mathsf{id} \otimes *_\theta)$

Proposition 4. *The family of connections* $\widetilde{\nabla} = P \circ (\mathsf{id} \otimes \mathsf{d})$ *is self-dual, i.e.*

$$(\mathsf{id} \otimes *_\theta) P((\mathsf{id} \otimes \mathsf{d})P)^2 = P((\mathsf{id} \otimes \mathsf{d})P)^2.$$

The projection P is equivalent to the projection $1 \otimes p$ in the algebra $M_4\left(\mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H})) \otimes \mathcal{A}(S^4_\theta)\right)$.

Define $V = (V_{ik}) \in M_4\left(\mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H})) \otimes \mathcal{A}(S^4_\theta)\right)$ by

$$V_{ik} = \rho^{-1} A_{ij} \otimes p_{jk} = \rho^{-1} A_{ij} \otimes \Psi_{ja}(\Psi^*)_{ak} = \rho^{-1} \widetilde{\psi}_{ia}(1 \otimes (\Psi^*)_{ak}),$$

then

$$(V^*V)_{il} = 1 \otimes p_{il} \qquad\qquad (VV^*)_{il} = P_{il},$$

the Chern characters $\mathsf{ch}_n(P) \in \mathsf{HC}_{2n}(\mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H})) \otimes \mathcal{A}(S^4_\theta))$

coincide with the pushforwards $\phi_* \mathsf{ch}_n(p)$ of $\mathsf{ch}_n(p) \in \mathsf{HC}_{2n}(\mathcal{A}(S^4_\theta))$ under the algebra map

$$\phi : \mathcal{A}(S^4_\theta) \to \mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H})) \otimes \mathcal{A}(S^4_\theta), \quad a \mapsto 1 \otimes a.$$

then $\mathsf{ch}_0(P) = \mathsf{ch}_1(P) = 0$, from the same for $\mathsf{ch}_0(p)$ and $\mathsf{ch}_1(p)$

Also, pull back the fundamental class $[S^4_\theta] \in \mathsf{HC}^4(\mathcal{A}(S^4_\theta))$ to a class $\phi^*[S^4_\theta]$ in $\mathsf{HC}^4(\mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H})) \otimes \mathcal{A}(S^4_\theta))$.

When paired with $\mathsf{ch}_2(P)$ it gives

$$\langle \mathsf{ch}_2(P), \phi^*[S^4_\theta] \rangle = \langle \phi_* \mathsf{ch}_2(p), \phi^*[S^4_\theta] \rangle = \langle \mathsf{ch}_2(p), [S^4_\theta] \rangle = 1$$

A family of connections

When transforming Ψ by the coaction of $\mathsf{SL}_\theta(2,\mathbb{H})$ in (3), one transforms the connection 1-form ω as well to $\widetilde{\omega} = (\widetilde{\omega}_{ab})$ with,

$$\widetilde{\omega}_{ab} := \Delta_L(\omega_{ab}) = \frac{1}{2}\sum (A^*)_{ik}A_{kj} \otimes \Big((\Psi^*)_{ai} \mathrm{d}\Psi_{jb} - \mathrm{d}(\Psi^*)_{ai}\Psi_{jb} \Big).$$

$\widetilde{\omega}$ still traceless ($\sum_a \widetilde{\omega}_{aa} = 0$) and skew-hermitian ($\widetilde{\omega}_{ab} = -\overline{\widetilde{\omega}}_{ba}$).

The instanton connection 1 form ω is left coinvariant under the coaction of the quantum group $\mathsf{Sp}_\theta(2)$, i.e. for it

$$\Delta_L(\omega_{ab}) = 1 \otimes \omega_{ab}$$

The relevant space that parametrize the connection one-forms is not $\mathsf{SL}_\theta(2,\mathbb{H})$ but rather the quotient of $\mathsf{SL}_\theta(2,\mathbb{H})$ by $\mathsf{Sp}_\theta(2)$

with π the natural map from $\mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H}))$ to $\mathcal{A}(\mathsf{Sp}_\theta(2))$, the algebra of the quotient to be the algebra of coinvariant of the natural left coaction $\Delta_R = (\pi \otimes \mathrm{id}) \circ \Delta$ of $\mathsf{Sp}_\theta(2)$ on $\mathsf{SL}_\theta(2,\mathbb{H})$:

$$\mathcal{A}(\mathcal{M}_\theta) := \{a \in \mathcal{A}(\mathsf{SL}_\theta(2,\mathbb{H})) \mid \Delta_R(a) = 1 \otimes a\}.$$

The quotient algebra $\mathcal{A}(\mathcal{M}_\theta)$ is well defined since $\mathsf{Sp}_\theta(2)$ is a quantum subgroup of $\mathsf{SL}_\theta(2,\mathbb{H})$: it is generated by the elements

$$m_{ij} := \sum_k (A^*)_{ik} A_{kj}$$

The moduli space of connections and its geometry

We will think of the transformed $\widetilde{\omega}$ as *a family of connection one-forms parametrized by the noncommutative space* $\mathcal{M}_\theta$:

$$\widetilde{\omega}_{ab} := \Delta_L(\omega_{ab}) = \frac{1}{2}\sum m_{ij} \otimes \Big((\Psi^*)_{ai}\mathrm{d}\Psi_{jb} - \mathrm{d}(\Psi^*)_{ai}\Psi_{jb}\Big).$$

At $\theta = 0$, for each point x in $\mathrm{SL}(2,\mathbb{H})$, the evaluation map $ev_x : \mathcal{A}(\mathrm{SL}(2,\mathbb{H})) \to \mathbb{C}$ gives an instanton connection (it has self-dual curvature) $(ev_x \otimes \mathrm{id})\widetilde{\omega}$ on the bundle $(ev_x \otimes \mathrm{id})P$ over S^4.

The structure of the algebra $\mathcal{A}(\mathcal{M}_\theta)$; collect the generators $m_{ij} = \sum_k (A^*)_{ik} A_{kj}$ into a matrix $M := (m_{ij})$

Explicitly,

$$M = \begin{pmatrix} m & 0 & g_1 & g_2^* \\ 0 & m & -\,\overline{\mu}\; g_2 & \mu\; g_1^* \\ g_1^* & -\,\mu\; g_2^* & n & 0 \\ g_2 & \overline{\mu}\; g_1 & 0 & n \end{pmatrix}$$

having commutation relations:

m, n are central : $\quad m\ x = x\ m, \quad n\ x = x\ n \quad \forall\, x \in \mathcal{M}_\theta$

g_1, g_2 are normal : $\quad g_1 g_1^* = g_1^* g_1, \quad g_2 g_2^* = g_2^* g_2$

and

$$g_1 g_2 = \mu^2 g_2 g_1, \quad g_1 g_2^* = \overline{\mu}^2 g_2^* g_1,$$
$$g_1^* g_2 = \overline{\mu}^2 g_2 g_1^*, \quad g_2^* g_1^* = \overline{\mu}^2 g_1^* g_2^*$$

With a quadratic relation:

$$mn - (g_1^* g_1 + g_2^* g_2) = 1 \qquad (4)$$

The boundary of the moduli space

The defining matrix M of $\mathcal{M}_\theta$, with the commutation relations among its entries is strikingly similar to the defining projection p of $\mathcal{A}(S^4_\theta)$ with the corresponding commutation relations.

The crucial difference is that while for $\mathcal{A}(S^4_\theta)$ we have a spherical relation, for $\mathcal{M}_\theta$ we have the relation (4) which make $\mathcal{M}_\theta$ a θ-deformation of a hyperboloid in 6 dimensions.

The hyperboloid structure is made evident by introducing two central elements w and y:

$$w := m + n; \qquad y := m - n.$$

giving

$$w^2 - (y^2 + g_1^* g_1 + g_2^* g_2) = 1,$$

making evident the hyperboloid structure.

The structure at ‘infinity’. First add the inverse of w to $\mathcal{A}(\mathcal{M}_\theta)$, and stereographically project onto the coordinates,

$$Y := w^{-1} y, \quad G_1 := w^{-1} g_1, \quad G_2 := w^{-1} g_2.$$

The hyperboloid relation becomes,

$$Y^2 + G_1^* G_1 + G_2^* G_2 = 1 - w^{-2}.$$

Evaluating w as a real number, and taking its ‘limit to infinity’ we get the spherical relation,

$$Y^2 + G_1^* G_1 + G_2^* G_2 = 1.$$

If we combine this with the commutation relations, we can conclude that at the 'boundary' of $\mathcal{M}_\theta$, we re-encounter the noncommutative 4-sphere $\mathcal{A}(S^4_\theta)$ via the identification

$$Y \leftrightarrow x\ , \qquad G_1 \leftrightarrow \alpha \qquad G_2 \leftrightarrow \beta.$$

This construction resembles the classical case, in which 4-spheres are found at the boundary of the moduli space.

Index theorem give a completeness argument:
the dimension of the "tangent space" to the moduli space is the index of a twisted Dirac operator coupled to the gauge field (and is computed with a nc local index formula)

to be continued ...

higher values of the topological charge