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Some $SU_q(2)$-bundles over quantum spheres

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First: on a symplectic quantum 4-sphere;
there is symmetry for $Sp_q(2)$
This comes from a quantum principal bundles with $SU_q(2)$ as structure group (a.k.a. Hopf-Galois extensions)

G.L., C. Pagani, C. Reina, C.M.P. 263 (2006) 65-88

Second : on a orthogonal quantum 4-sphere;
there is symmetry for $SO_q(5)$
No principal bundle (yet ?)

F. D'Andrea, L. Dabrowski, G.L., CMP. in press

The symplectic groups $A(Sp_q(n))$:

the Hopf algebra generated by matrix elements T_i^j's with commutation rules coming from the R matrix of the C-series

Have comodule-subalgebras $A(S_q^{4n-1})$ yielding quantum homogeneous spaces [BM]

The relevant case is $n = 2$:
$A(S_q^7)$ is the quantum version of the homogeneous space $Sp(2)/Sp(1)$;
the injection $A(S_q^7) \hookrightarrow A(Sp_q(2))$ is a quantum principal bundle with "structure Hopf algebra" $A(Sp_q(1))$

In turn, S^7_q is the total space of a quantum $SU_q(2)$ principal bundle over a quantum 4-sphere S^4_q

This is not a quantum homogeneous structure.

The algebra $A(S^4_q)$ is the subalgebra of $A(S^7_q)$ generated by the matrix elements of a self-adjoint projection $p = vv^*$ with v a 4×2 matrix whose entries generators of $A(S^7_q)$.

The naive choice: extra generators which vanish at $q = 1$.

An alternative twisted v yields the right number of generators of an algebra deforming the polynomial algebra of S^4.

This good choice becomes even better: a natural coaction of $SU_q(2)$ on $A(S^7_q)$ with $A(S^4_q)$ as algebra of coinvariant elements

$A(S^4_q) \hookrightarrow A(S^7_q)$ a non trivial $A(SU_q(2))$-principal bundle

$A(Sp_q(N,\mathbb{C}))$ he associative nc algebra generated by the entries $T_i^{\,j}$, $i,j=1,\ldots,N$ of a matrix T which satisfies RTT equations:

$$R\,T_1T_2 = T_2T_1R\ , \qquad\qquad T_1 = T\otimes 1\ , \quad T_2 = 1\otimes T\ .$$

The relevant matrix R is the one for the C_N series [FRT],

$$R = q\sum_{i=1}^{N} e_i{}^i\otimes e_i{}^i + \sum_{\substack{i,j=1\\ i\neq j,j'}}^{N} e_i{}^i\otimes e_j{}^j + q^{-1}\sum_{i=1}^{N} e_{i'}{}^{i'}\otimes e_i{}^i$$
$$+\,(q-q^{-1})\sum_{\substack{i,j=1\\ i>j}}^{N} e_i{}^j\otimes e_j{}^i - (q-q^{-1})\sum_{\substack{i,j=1\\ i>j}}^{N} q^{\rho_i-\rho_j}\varepsilon_i\varepsilon_j\ e_i{}^j\otimes e_{i'}{}^{j'}$$

The symplectic group structure comes by imposing the additional

$$TCT^tC^{-1} = CT^tC^{-1}T = 1$$

with the matrix ${C_i}^j = q^{\rho_j}\varepsilon_i\delta_{ij'}$

The Hopf algebra co-structures of the quantum group $Sp_q(N,\mathbb{C})$:

$$\Delta(T) = T \dot{\otimes} T \ , \quad \varepsilon(T) = I \ , \quad S(T) = CT^tC^{-1} \ .$$

The compact real form $A(Sp_q(n))$ of the quantum group $A(Sp_q(N,\mathbb{C}))$ is given by taking $q \in \mathbb{R}$ and the anti-involution

$$\overline{T} = S(T)^t = C^tT(C^{-1})^t$$

Elements : $\quad x_i = T_i{}^N \;, \quad v^j = S(T)_N{}^j \;, \quad i,j = 1,\ldots,N \;.$
generate subalgebras of $A(Sp_q(N,\mathbb{C}))$.

With the previous involution: $v^i = S(T)_N{}^i = \bar{x}^i$.

The subalgebra $A(S_q^{4n-1})$ of $A(Sp_q(n))$ generated by the elements $\{x_i, v^i = \bar{x}^i, \; i = 1,\ldots,2n\}$ is the algebra of polynomial functions on a sphere:

$S(T)T = I \;\Rightarrow\; \sum S(T)_N{}^i T_i{}^N = \delta_N^N = 1$, i.e. $\sum_i \bar{x}^i x_i = 1$.

The restriction of the comultiplication is a natural left coaction

$$\Delta_L : A(S_q^{4n-1}) \longrightarrow A(Sp_q(n)) \otimes A(S_q^{4n-1}) \;.$$

and $A(S_q^{4n-1})$ is a comodule algebra for $A(Sp_q(n))$.

The symplectic 7-sphere S_q^7 : with the relation, $\sum_i \bar{x}^i x_i = 1$, also

$$
\begin{aligned}
&x_1 x_2 = q x_2 x_1 \,, && x_1 x_3 = q x_3 x_1 \,, \\
&x_2 x_4 = q x_4 x_2 \,, && x_3 x_4 = q x_4 x_3 \,, \\
&x_4 x_1 = q^{-2} x_1 x_4 \,, && x_3 x_2 = q^{-2} x_2 x_3 + q^{-2}(q^{-1} - q) x_1 x_4 \,,
\end{aligned}
$$

$$
\begin{aligned}
&x_1 \bar{x}^1 = \bar{x}^1 x_1 \,, && x_1 \bar{x}^2 = q^{-1} \bar{x}^2 x_1 \,, \\
&x_1 \bar{x}^3 = q^{-1} \bar{x}^3 x_1 \,, && x_1 \bar{x}^4 = q^{-2} \bar{x}^4 x_1 \,,
\end{aligned}
$$

$$
\begin{aligned}
x_2 \bar{x}^2 &= \bar{x}^2 x_2 + (1 - q^{-2}) \bar{x}^1 x_1 \,, \\
x_2 \bar{x}^3 &= q^{-2} \bar{x}^3 x_2 \,, \\
x_2 \bar{x}^4 &= q^{-1} \bar{x}^4 x_2 + q^{-1}(q^{-2} - 1) \bar{x}^3 x_1 \,,
\end{aligned}
$$

$$
\begin{aligned}
x_3 \bar{x}^3 &= \bar{x}^3 x_3 + (1 - q^{-2})[\bar{x}^1 x_1 + (1 + q^{-2}) \bar{x}^2 x_2] \,, \\
x_3 \bar{x}^4 &= q^{-1} \bar{x}^4 x_3 + (1 - q^{-2}) q^{-3} \bar{x}^2 x_1 \,,
\end{aligned}
$$

$$
x_4 \bar{x}^4 = \bar{x}^4 x_4 + (1 - q^{-2})[(1 + q^{-4}) \bar{x}^1 x_1 + \bar{x}^2 x_2 + \bar{x}^3 x_3] \,,
$$

$A(S^7_q)$ is the subalgebra of $A(Sp_q(2))$ made of coinvariants under the right-coaction of $A(Sp_q(1))$

Proposition 1. *The two-sided *-ideal in $A(Sp_q(2))$ generated as*

$$I_q = \{T_1{}^1 - 1, T_4{}^4 - 1, T_1{}^2, T_1{}^3, T_1{}^4, T_2{}^1, T_2{}^4, T_3{}^1, T_3{}^4, T_4{}^1, T_4{}^2, T_4{}^3\}$$

is a Hopf ideal. The Hopf algebra $B_q := A(Sp_q(2))/I_q$ is isomorphic to the coordinate algebra $A(SU_{q^2}(2)) \cong A(Sp_q(1))$.

Proposition 2. *The algebra $A(S^7_q) \subset A(Sp_q(2))$ is the algebra of coinvariants with respect to the natural right coaction*

$$\Delta_R : A(Sp_q(2)) \to A(Sp_q(2)) \dot{\otimes} A(Sp_q(1)) \quad ; \quad \Delta_R(T) = T \dot{\otimes} T' \,.$$

The previous is an example of a quantum principal bundle over a quantum homogeneous space [BM]:

The latter is the datum of a Hopf quotient $\pi : A(G) \to A(K)$ with the right coaction of $A(K)$ on $A(G)$ given by the reduced coproduct $\Delta_R := (id \otimes \pi)\Delta$ where Δ is the coproduct of $A(G)$.

The subalgebra $B \subset A(G)$ made of the coinvariants with respect to Δ_R is a quantum homogeneous space. With some minor additional assumptions it is a quantum principal bundle.

The fundamental step is to make the S^7_q itself into the total space of a quantum principal bundle over a deformed 4-sphere.

This is not a quantum homogeneous space construction; it is not obvious that such a bundle exists at all.

Quantum principle bundles
(also known as of Hopf-Galois extensions) [BM,HM]

Definition 3. *Let H be a Hopf algebra and P a right H-comodule algebra with multiplication $m : P \otimes P \to P$ and coaction $\Delta_R : P \to P \otimes H$. Let $B \subseteq P$ be the subalgebra of coinvariants, i.e. $B = \{p \in P \mid \Delta_R(p) = p \otimes 1\}$. The extension $B \subseteq P$ is called an H Hopf-Galois extension if the canonical map*

$$\chi : P \otimes_B P \longrightarrow P \otimes H \ ,$$
$$\chi := (m \otimes id) \circ (id \otimes_B \Delta_R) \ , \quad p' \otimes_B p \mapsto \chi(p' \otimes_B p) = p' p_{(0)} \otimes p_{(1)}$$

is bijective.

$$\Delta_R(p) = p_{(0)} \otimes p_{(1)}$$

Injectivity of the canonical map dualizes the condition of a group action $X \times G \to X$ to be free:

if $\alpha : X \times G \to X \times_M X, \ (x, g) \mapsto (x, x \cdot g)$ then $\alpha^* = \chi$
with P, H the algebras of functions on X, G respectively and the action is free if and only if α is injective.

$M := X/G$ the space of orbits with projection $\pi : X \to M$

For our examples, thanks to the fact that the "structure group" is $SU_q(2)$, further nice properties can be established [Sc]

On the free module $\mathcal{E} := \mathbb{C}^4 \otimes A(S_q^7)$ the hermitean structure:

$$h(|\xi_1\rangle, |\xi_2\rangle) = \sum_{j=1}^{4} \bar{\xi_1}^j \xi_2^j \,.$$

To $|\xi\rangle \in \mathcal{E}$, associate $\langle\xi|$ in the dual module $\mathcal{E}^*$ by the pairing

$$\langle\xi| (|\eta\rangle) := \langle\xi|\eta\rangle = h(|\xi\rangle, |\eta\rangle).$$

Look for two elements $|\phi_1\rangle, |\phi_2\rangle$ in $\mathcal{E}$ s.t.

$$\langle\phi_1|\phi_1\rangle = 1 \,, \quad \langle\phi_2|\phi_2\rangle = 1 \,, \quad \langle\phi_1|\phi_2\rangle = 0 \,.$$

Then, the matrix valued function

$$p := |\phi_1\rangle \langle\phi_1| + |\phi_2\rangle \langle\phi_2| \,,$$

is a self-adjoint idempotent (a projection).

In principle, $p \in \mathrm{Mat}_4(A(S^7_q))$; choose $|\phi_1\rangle$, $|\phi_2\rangle$ s.t. the entries of p generate a subalgebra $A(S^4_q)$ of $A(S^7_q)$ which deforms the algebra of polynomial functions on the 4-sphere S^4.

A good choice is

$$|\phi_1\rangle = (q^{-3}x_1, -q^{-1}\bar{x}^2, q^{-1}x_3, -\bar{x}^4)^t \ ,$$
$$|\phi_2\rangle = (q^{-2}x_2, q^{-1}\bar{x}^1, -x_4, -\bar{x}^3)^t$$

The matrix

$$v = (|\phi_1\rangle \, , |\phi_2\rangle) = \begin{pmatrix} q^{-3}x_1 & q^{-2}x_2 \\ -\, q^{-1}\bar{x}^2 & q^{-1}\bar{x}^1 \\ q^{-1}x_3 & -\, x_4 \\ -\, \bar{x}^4 & -\, \bar{x}^3 \end{pmatrix}$$

satisfy $v^* v = 1$ and hence $p = vv^*$ is a self-adjoint projection.

Explicitly,

$$p = \begin{pmatrix} q^{-2}t & 0 & a & b \\ 0 & t & q^{-2}\overline{b} & -q^{2}\overline{a} \\ \overline{a} & q^{-2}b & 1-q^{-4}t & 0 \\ \overline{b} & -q^{2}a & 0 & 1-q^{2}t \end{pmatrix} .$$

with

$$t = q^{-2}\bar{x}^{2}x_{2} + q^{-2}\bar{x}^{1}x_{1} \ ,$$
$$a = q^{-4}x_{1}\bar{x}^{3} - q^{-2}x_{2}\bar{x}^{4} \ , \qquad b = -q^{-3}x_{1}x_{4} - q^{-2}x_{2}x_{3}$$

Moreover, $\overline{t} = t$, and $\overline{a}, \overline{b}$ the conjugate to a, b respectively.

They satisfy

$$ab = q^4 ba \;, \qquad \bar{a}b = b\bar{a} \;,$$
$$ta = q^{-2} at \;, \quad tb = q^4 bt \;,$$

and sphere relations

$$a\bar{a} + b\bar{b} = q^{-2} t(1 - q^{-2} t) \;, \qquad q^4 \bar{a}a + q^{-4} \bar{b}b = t(1 - t) \;,$$
$$b\bar{b} - q^{-4} \bar{b}b = (1 - q^{-4}) t^2 \;.$$

Also,

$$q^{-2} p_{11} + q^2 p_{22} + p_{33} + p_{44} = 2$$

a deformation of the algebras of polynomial functions $A(S^4)$

The $SU_q(2)$-coaction

A coaction of the quantum group $SU_q(2)$ on $A(S_q^7)$ s.t. the algebra $A(S_q^4)$ is the algebra of coinvariants.

The defining matrix of the quantum group $SU_q(2)$:

$$\begin{pmatrix} \alpha & -q\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix}$$

with commutation relations,

$$\begin{array}{lll} \alpha\gamma = q\gamma\alpha \ , & \alpha\bar{\gamma} = q\bar{\gamma}\alpha \ , & \gamma\bar{\gamma} = \bar{\gamma}\gamma \ , \\ \alpha\bar{\alpha} + q^2\bar{\gamma}\gamma = 1 \ , & \bar{\alpha}\alpha + \bar{\gamma}\gamma = 1 \ . & \end{array}$$

A coaction of $SU_q(2)$ on the matrix v by,

$$\delta_R(v) := \begin{pmatrix} q^{-3}x_1 & q^{-2}x_2 \\ -q^{-1}\bar{x}^2 & q^{-1}\bar{x}^1 \\ q^{-1}x_3 & -x_4 \\ -\bar{x}^4 & -\bar{x}^3 \end{pmatrix} \dot{\otimes} \begin{pmatrix} \alpha & -q\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix} .$$

The entries of $p = vv^*$ are automatically coinvariants

δ_R extends as an algebra homomorphism to the whole of $A(S^7_q)$.

Proposition 4. *The coaction δ_R is a right coaction of the quantum group $SU_q(2)$ on the 7-sphere S^7_q,*

$$\delta_R : A(S^7_q) \longrightarrow A(S^7_q) \otimes A(SU_q(2)) .$$

The algebra $A(S^4_q)$ is the algebra of coinvariants.

Representations of the algebra $A(S_q^4)$; restrict to $|q| < 1$

The representation β: is the trivial one

$$t = 0, \quad a = 0, \quad b = 0 \ ,$$

the representation Hilbert is just $\mathbb{C}$; of course, $\beta(1) = 1$.

The representation σ: is infinite dimensional.

$$\begin{aligned}
t\,|m,n\rangle &= q^{2m+4n+4}\,|m,n\rangle \ , \\
a\,|m,n\rangle &= (1-q^{2m})^{\frac{1}{2}} q^{m+2n}\,|m-1,n\rangle \ , \\
b\,|m,n\rangle &= (1-q^{4n+4})^{\frac{1}{2}} q^{2(m+n+2)}\,|m,n+1\rangle \ ,
\end{aligned}$$

The algebra generators are all trace class, in particular,

$$\mathsf{Tr}(t) = q^4 \sum_m q^{2m} \sum_n q^{4n} = \frac{q^4}{(1-q^2)(1-q^4)}$$

The closure of $A(S_q^4)$ is the C^*-algebra $\mathcal{C}(S_q^4) = \mathcal{K} \oplus \mathbb{C}\mathbb{I}$.

The index pairings

The 'defining' projection p determines a class $[p] \in K_0[\mathcal{C}(S_q^4)]$

A way to prove its nontriviality is by pairing it with a nontrivial element in the dual K-homology, i.e. with (the class of) a nontrivial Fredholm module $[\mu] \in K^0[\mathcal{C}(S_q^4)]$.

Use the pairing of the corresponding the Chern characters in the cyclic homology $\mathrm{ch}_*(p) \in HC_*[A(S_q^4)]$ and cyclic cohomology $\mathrm{ch}^*(\mu) \in HC^*[A(S_q^4)]$

Enough to consider $HC_0[A(S_q^4)]$ and dually to take a suitable trace of the projector.

A component in degree zero, $\mathrm{ch}_0(p) \in HC_0[A(S^4_q)]$:

$$\mathrm{ch}_0(p) := \mathrm{tr}(p) = 2 - q^{-4}(1-q^2)(1-q^4)\ t \ \in A(S^4_q)$$

The K-homology as homotopy classes of Fredholm modules.

A 0^+-summable Fredholm module $[\mu] \in K^0[\mathcal{C}(S^4_q)]$;

the analogous element of $K_0(S^4)$ for the undeformed sphere is a 4^+-summable Fredholm module, the fundamental class of S^4.

The Fredholm module $\mu := (\mathcal{H}, \Psi, \gamma)$:
the Hilbert space is $\mathcal{H} = \mathcal{H}_\sigma \oplus \mathcal{H}_\sigma$
the representation is $\Psi = \sigma \oplus \beta$ with β trivially extended to $\mathcal{H}_\sigma$
the grading operator is

$$\gamma = \mathrm{diag}(1, -1).$$

The corresponding Chern character $\mathsf{ch}^*([\mu])$ has a component in degree 0, $\mathsf{ch}^0(\mu) \in HC^0[A(S_q^{2n})]$: is the trace

$$\tau^1(x) := \mathsf{Tr}\,(\gamma\Psi(x)) = \mathsf{Tr}\,(\sigma(x) - \beta(x))\,.$$

The operator $\sigma(x) - \beta(x)$ is always trace class.

The pairing:

$$\begin{aligned}\langle [\mu], [p] \rangle &:= \Big\langle \mathsf{ch}^0(\mu), \mathsf{ch}_0(p) \Big\rangle \\ &= -q^{-4}(1-q^2)(1-q^4)\,\mathsf{Tr}(t) \\ &= -q^{-4}(1-q^2)(1-q^4)q^4(1-q^2)^{-1}(1-q^4)^{-1} \\ &= -1\ .\end{aligned}$$

This shows that the right $A(S_q^4)$-module $p[A(S_q^4)^4]$ is not free

The ‘trivial’ element in $K^0[\mathcal{C}(S^4_q)]$ measure the ‘rank’ of p.

The trivial generator of the K-homology $K_0(S^4)$ of the classical sphere S^4 is the image of the generator of the K-homology of a point by the functorial map $K_*(\iota) : K_0(*) \to K_0(S^N)$, where $\iota : * \hookrightarrow S^N$ is the inclusion of a point into the sphere.

The quantum sphere S^4_q has just one ‘classical point’, i.e. the 1-dimensional representation β.

The corresponding Fredholm module $[\varepsilon] \in K^0[\mathcal{C}(S^4_q)]$:

the Hilbert space is $\mathbb{C}$ with representation β;
the grading operator is $\gamma = 1$.

The component $\mathrm{ch}^0(\varepsilon) \in HC^0[A(S_q^{2n})]$ of the corresponding Chern character is the trace given by the representation

$$\tau^0(x) = \beta(x) \ ,$$

and vanishes on all the generators, whereas $\tau^0(1) = 1$.

Not surprisingly,

$$\langle [\varepsilon], [p] \rangle := \tau^0(\mathrm{ch}_0(p)) = \beta(2) = 2 \ .$$

Quantum principal bundle structure

H a Hopf algebra; P a right H-comodule algebra with multiplication $m : P \otimes P \to P$ and coaction $\Delta_R : P \to P \otimes H$ and $B \subseteq P$ is the subalgebra of coinvariants. The extension $B \subseteq P$ is H Hopf-Galois if the canonical map

$$\chi : P \otimes_B P \longrightarrow P \otimes H \,, p' \otimes_B p \mapsto \chi(p' \otimes_B p) = p' p_{(0)} \otimes p_{(1)} \,,$$

is bijective.

For quantum structure groups which are cosemisimple and have bijective antipodes, as for $SU_q(2)$, things are easy [SC]:

1. the surjectivity of the canonical map implies bijectivity and 'faithfully flatness'

2. the map χ is surjective whenever, for any generator h of H, the element $1 \otimes h$ is in its image

Proposition 5. *The extension* $A(S_q^4) \subset A(S_q^7)$ *is* $A(SU_q(2))$-*quantum principal bundle.*

Proof. The crucial result is that

$$\chi \begin{pmatrix} \left\langle \phi_1 \dot{\otimes}_{A(S_q^4)} \phi_1 \right\rangle & \left\langle \phi_1 \dot{\otimes}_{A(S_q^4)} \phi_2 \right\rangle \\ \left\langle \phi_2 \dot{\otimes}_{A(S_q^4)} \phi_1 \right\rangle & \left\langle \phi_2 \dot{\otimes}_{A(S_q^4)} \phi_2 \right\rangle \end{pmatrix} = 1 \dot{\otimes} \begin{pmatrix} \alpha & -q\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix} ,$$

with the definition

$$\left\langle \xi_1 \dot{\otimes}_B \xi_2 \right\rangle := \sum_{j=1}^{m} \bar{\xi}_1^j \otimes_B \xi_2^j \, .$$

Associated bundles and coequivariant maps

$B \subset P$ a H-Galois extension with Δ_R the coaction of H on P.

$\rho : V \to H \otimes V$ be a corepresentation of H with V a finite dimensional vector space, and S is the antipode of H.

A coequivariant map is any $\varphi \in P \otimes V$ s.t.

$$(\Delta_R \otimes id)\varphi = (id \otimes (S \otimes id) \circ \rho))\varphi\,;$$

The coequivariant make up a right and left B-module $\Gamma_\rho(P, V)$.

For our $A(SU_q(2))$-principal bundle $A(S^4_q) \subset A(S^7_q)$, $\rho_1 : \mathbb{C}^2 \to \mathbb{C}^2 \otimes A(SU_q(2))$ the fundamental corepresentation of $A(SU_q(2))$, with $\Gamma_1(A(S^7_q), \mathbb{C}^2)$ the $A(S^4_q)$-module of corresponding coequivariant maps.

Proposition 6. *The modules $\mathcal{E} := p[A(S^4_q)^4]$ and $\Gamma_1(A(S^7_q), \mathbb{C}^2)$ are isomorphic as right $A(S^4_q)$-modules.*

Proposition 7. *The $A(SU_q(2))$-principal bundle $A(S^4_q) \subset A(S^7_q)$ is not trivial.*

Proof. Triviality would imply that all modules of coequivariant maps are free.

But the module $p[A(S^4_q)^4] \simeq \Gamma_1(A(S^7_q), \mathbb{C}^2)$ is not free.

The QUE algebra $U_q(so(5))$:

generators $\{K_i = K_i^*, K_i^{-1}, E_i, F_i := E_i^*\}_{i=1,2}$ with relations

$$[K_1, K_2] = 0 \ , \qquad K_i K_i^{-1} = K_i^{-1} K_i = 1 \ ,$$
$$[E_i, F_j] = \delta_{ij} \frac{K_j^2 - K_j^{-2}}{q^j - q^{-j}} \ ,$$
$$K_i E_i K_i^{-1} = q^i E_i \ , \qquad K_i E_j K_i^{-1} = q^{-1} E_j \ \text{if } i \neq j \ ,$$

plus the conjugated ones and two Serre relations

The coalgebra structure:

$$\Delta K_i = K_i \otimes K_i \ , \quad \Delta E_i = E_i \otimes K_i + K_i^{-1} \otimes E_i$$
$$\epsilon(K_i) = 1 \ , \quad \epsilon(E_i) = 0$$
$$S(K_i) = K_i^{-1} \ , \quad S(E_i) = -q^i E_i$$

Irreducible rep.s of $U_q(so(5))$ [A. Chakrabarti]

$V_{(n_1,n_2)}$, $n_2 \in \frac{1}{2}\mathbb{N}$ and $n_2 - n_1 \in \mathbb{N}$,

the highest weight of $V_{(n_1,n_2)}$ is (n_1, n_2);
the components are eigenvalues of K_1 and K_1K_2

$V_l := V_{(\frac{1}{2},l)}$ if $l \in \mathbb{N} + \frac{1}{2}$; *spin* representations

$V_l := V_{(0,l)}$ if $l \in \mathbb{N}$; *vector* representations

the vector rep.s are *real*; the antilinear map $C : V_l \to V_l$

$$C\,|l, m_1, m_2; j\rangle := (-q)^{m_1} q^{3m_2}\,|l, -m_1, -m_2; j\rangle$$

satisfies $C^2 = 1$ and $ChC = S(h)^*$ for all $h \in U_q(so(5))$.

The quantum Euclidean 4-Sphere

Classically: generators of $A(S^4)$ are a linear basis for the rep. V_1;

The tensor $*$-algebra generated by V_1, modulo the two-sided $*$-ideal generated by $V_1 \wedge V_1$, is:

• commutative, since $V_1 \wedge V_1$ is spanned by the commutators;
• an $\mathcal{U}(so(5))$-module $*$-algebra, since $V_1 \wedge V_1 \subset V_1 \otimes V_1$ carries a (real) subrepr. and the ideal it generates is $\mathcal{U}(so(5))$-invariant.

The quotient is $A(\mathbb{R}^5)$. The radius

$$r^2 := x_0^2 + x_1 x_1^* + x_2 x_2^*$$

is central and $U_q(so(5))$-invariant (it is a basis of the scalar rep.). The quotient is identified with $A(S^4)$

$$A(S^4) = A(\mathbb{R}^5) / \left\langle r^2 - 1 \right\rangle$$

The same picture holds in the q-deformed case.

Let $\{x_0, x_1, x_2, x_1^*, x_2^*\}$ be a linear basis of V_1

We look for a $*$-algebra module: the action of $U_q(so(5))$ on x_i^* is determined by $h \triangleright x_i^* = \{S(h)^* \triangleright x_i\}^*$ (the antilinear map C is implemented by the involution)

choose the following normalization (up to a global constant)

$$x_2 \sim |1,0,1;0\rangle \ , x_1 \sim |1,1,0;1\rangle \ , \quad x_0 \sim (q[2])^{-1/2} |1,0,0;1\rangle$$

Then

$$x_1^* \sim C\,|1,0,1;0\rangle = -q\,|1,-1,0;1\rangle \ , \quad x_2^* \sim C\,|1,1,0;1\rangle = q^3\,|1,0,-1;0\rangle$$

the action of $U_q(so(5))$ on $\{x_i, x_i^*\}$ is uniquely determined

The quotient of the free algebra generated by $\{x_i, x_i^*\}$ by the ideal generated by $V_{(1,1)}$ is the quantum Euclidean space $A(\mathbb{R}_q^5)$.

the central radius

$$(1+q^{-6})r^2 := (1+q^2)x_0^2 + x_1x_1^* + x_2x_2^* + q^{-2}x_1^*x_1 + q^{-6}x_2^*x_2$$

in $A(\mathbb{R}_q^5)$ and $U_q(so(5))$-invariant (it spans the 1-dim. rep. V_0)

Modulo a global rescaling of the generators, there is only one quantum 4-sphere:

$$A(S_q^4) := A(\mathbb{R}_q^5)/\left\langle r^2 - 1 \right\rangle.$$

By construction an action of $U_q(so(5))$ on $A(S_q^4)$

Definition 8. *The algebra* $A(S^4_q)$ *of polynomial functions on the* quantum Euclidean 4-sphere *is generated by* $x_0 = x_0^*$, x_1, x_2, x_1^*, x_2^*, *with relations:*

$$x_i x_j = q^2 x_j x_i \ , \qquad \forall \ 0 \leq i < j \leq 2 \ ,$$
$$x_i^* x_j = q^2 x_j x_i^* \ , \qquad \forall \ i \neq j \ ,$$
$$[x_1^*, x_1] = (1 - q^4) x_0^2 \ , \qquad [x_2^*, x_2] = x_1^* x_1 - q^4 x_1 x_1^* \ ,$$
$$x_0^2 + x_1 x_1^* + x_2 x_2^* = 1 \ .$$

It is by construction an $U_q(so(5))$-module $*$-algebra.

An S^1-worth of classical points:

$$\sigma(x_0) = \sigma(x_1) = 0, \qquad \sigma(x_2) = e^{2\pi i \theta}$$

Proposition 9. *The defining relations of the algebra* $A(S_q^4)$ *are equivalent to the matrix-valued function*

$$P = \frac{1}{2}\begin{pmatrix} 1+q^2x_0 & qx_1 & x_2 & 0 \\ qx_1^* & 1-qx_0 & 0 & x_2 \\ x_2^* & 0 & 1-qx_0 & -x_1 \\ 0 & x_2^* & -x_1^* & 1+x_0 \end{pmatrix}$$

be a projection, i.e. $P^* = P$ *and* $P^2 = P$*.*

$$\mathrm{Tr}(P) = 2 + (q-1)^2 x_0$$

Equivariant representations of $\mathcal{A}(S_q^4)$
Equivariance:
a representation of the crossed product algebra $\mathcal{A}(S_q^4) \rtimes U_q(so(5))$,

$$ha = (h_{(1)} \triangleright a) h_{(2)}, \qquad \forall a \in \mathcal{A}(S_q^4), \quad h \in U_q(so(5))$$

The left regular representation on

$$\mathcal{H}_0 = L^2(S_q^4) = \Big(\mathcal{A}(S_q^4) \simeq \bigoplus_{l \in \mathbb{N}} V_l \Big)^{cl}$$

Two spin representations on

$$\mathcal{H}_\pm = \Big(\bigoplus_{l \in \mathbb{N} + \frac{1}{2}} V_l \Big)^{cl}$$

Proposition 10.
Take $\mathcal{H} := \mathcal{H}_+ \oplus \mathcal{H}_-$*, with natural grading* γ*, and* $F \in \mathcal{B}(\mathcal{H})$*,*

$$F\,|l, m_1, m_2; j\rangle_\pm := |l, m_1, m_2; j\rangle_\mp \ ,$$

the data $(\mathcal{A}(S^4_q), \mathcal{H}, F, \gamma)$
is a 1^+*-summable not trivial Fredholm module.*

In particular,

$$\left\langle \mathrm{ch}^F, [P] \right\rangle = 1 \ ,$$

with P *the defining idempotent*

The orthogonal sphere S_q^4 has an isospectral even spectral triple

It is real up to "infinitesimals'

The Dirac operator:
Proposition 11. *Let D be the (unbounded) operator on $\mathcal{H}$:*

$$D\,|l, m_1, m_2; j\rangle_{\pm} := (l + \tfrac{3}{2})\,|l, m_1, m_2; j\rangle_{\mp} \ .$$

Then, $(\mathcal{A}(S_q^4), \mathcal{H}, D, \gamma)$ is a regular equivariant 4^+-summable even spectral triple. It has not-trivial K-homology

D is isospectral to the classical Dirac operator on S^4.
When $q = 1$, the above spectral triple is the canonical one of the round metric of S^4

Dimension spectrum and the top residue

Proposition 12. *The dimension spectrum* Σ *satisfies*

$$\Sigma \cap \{\mathrm{Re}\, s > 2\} = \{3, 4\},$$

and in such an half plane is made of simple poles only. The top residue coincides with the Lebesgue integral on the subspace of classical points of S_q^4*:*

$$⨍ a|D|^{-4} = \frac{4}{3}\int_0^1 \sigma(a)\mathrm{d}\theta \ ,$$

with $\sigma : \mathcal{A}(S_q^4) \to C(S^1)$ *the* $*$*-algebra morphism*

$\sigma(x_0) = \sigma(x_1) = 0$ *and* $\sigma(x_2) = e^{2\pi i\theta}$

Real structure

The antilinear operator

$$J\,|l, m_1, m_2; j\rangle_\pm = i^{2l+1}(-1)^{j+m_1}\,|l, -m_1, -m_2; j\rangle_\pm$$

is the antiunitary part of the equivariant antilinear operator T

$$T\,|l, m_1, m_2; j\rangle_\pm = i^{2l+1}(-1)^{j+m_1} q^{m_1+3m_2}\,|l, -m_1, -m_2; j\rangle_\pm \ .$$

equivariance is that $Th = S(h)^* T$ for all $h \in U_q(so(5))$.

J is such that

$J^2 = -1$	(antiunitarity)
$J\gamma = \gamma J$	(parity)
$JD = DJ$	(compatibility with D)

Finally,

Proposition 13. *Let J be the above antilinear isometry. Then,*

$$[a, JbJ] \in \mathsf{OP}^{-\infty} ,$$

$$[[D,a], JbJ] \in \mathsf{OP}^{-\infty}, \qquad \forall\, a,b \in \mathcal{A}(S^4_q) ,$$

with $\mathsf{OP}^{-\infty}$ the (two-sided) ideal of smoothing operators,

$$\mathsf{OP}^{-\infty} := \{T \in \mathsf{OP}^0 \,:\, |D|^n T \in \mathsf{OP}^0 \ \ , \forall\, n \in \mathbb{N}\}$$

$\mathsf{OP}^0 := \bigcap_{j\in\mathbb{N}} \mathrm{dom}\, \delta^j, \qquad \delta(a) = [|D|, a]$.