Graded Algebras

$\mathbb{K}$ algebraically closed of characteristic $= 0$

$\mathcal{A} \in \mathbf{GrAlg} \Leftrightarrow \mathcal{A} = T(E)/I$ with $\dim(E) < \infty$ and I generated by a finite number of homogeneous elements of degrees ≥ 2.

$\Rightarrow \mathcal{A}$ connected graded $\mathbb{K}$-algebra finitely generated in degree $=1$.

Generators-relations : $E = \oplus_{\lambda=1}^{d} \mathbb{K} x^{\lambda}$, $I = (f_1, \dots, f_r)$
$\mathcal{A} = \mathbb{K}\langle x^1, \dots, x^d \rangle / (f_1, \dots, f_r)$
$f_\alpha \in E^{\otimes^{N_\alpha}}$, $N_\alpha \geq 2$, $r=$ minimum nb.

$\mathcal{A} \in \mathbf{H_N Alg}$ if $I = (R)$ with $R \subset E^{\otimes^N}$
$(N_\alpha = N,\ \forall \alpha \in \{1, \dots, r\})$.

Global Dimension

Work with $\mathcal{A} \in \mathbf{GrAlg}$ and graded modules

$\mathcal{A} = \mathbb{K}\langle x^1, \dots, x^d\rangle/(f_1, \dots, f_r)$
$f_\alpha = M_{\alpha\lambda} x^\lambda$ with $M_{\alpha\lambda} \in E^{\otimes^{N_\alpha - 1}} \Rightarrow$
$\mathcal{A}^r \stackrel{M}{\to} \mathcal{A}^d \stackrel{x}{\to} \mathcal{A} \stackrel{\varepsilon}{\to} \mathbb{K} \to 0.$
exact by def., can be extended as minimal projective resolution of the left $\mathcal{A}$-module $\mathbb{K}$
$\to M_n \to \cdots \to M_0 \to \mathbb{K} \to 0$
$M_0 = \mathcal{A}$, $M_1 = \mathcal{A}^d$, $M_2 = \mathcal{A}^r$

$\mathcal{A}$ has finite global dimension D,
gldim$(\mathcal{A}) = D \in \mathbb{N}$, iff. one has
$0 \to M_D \to \cdots \to M_0 \to \mathbb{K} \to 0$
with $M_D \neq 0$, (otherwise gldim$(\mathcal{A}) = \infty$).
One has also :
gldim$(\mathcal{A})$=Hochschild dim$(\mathcal{A})$.

The M_n can be assumed to be free.

Gorenstein Algebras

$\mathcal{A} = \mathbb{K}\langle x^1, \dots, x^d\rangle/(f_1, \dots, f_r)$,
$\text{gldim}(\mathcal{A}) = D < \infty$
$0 \to M_D \to \cdots \to M_0 \to \mathbb{K} \to 0$
as above. Apply $\text{Hom}_{\mathcal{A}}(\bullet, \mathcal{A})$ to the complex
$0 \to M_D \to \cdots \to M_0 \to 0 \Rightarrow$
complex M' of right $\mathcal{A}$-modules
$0 \to M'_0 \to \cdots \to M'_D \to 0$

$\mathcal{A}$ is *Gorenstein* iff
$H^n(M') = 0$ for $n < D$ and $H^D(M') = \mathbb{K}$
i.e. iff one has a resolution (min. proj.)
$0 \to M'_0 \to \cdots \to M'_D \to \mathbb{K} \to 0$
of the right $\mathcal{A}$-module $\mathbb{K}$.
$\sim$ *Poincaré duality*.

Regular Algebras

$\mathcal{A} \in \mathbf{GrAlg}$ is *regular* iff.
gldim$(\mathcal{A}) = D < \infty$ and $\mathcal{A}$ is Gorenstein.

<u>Remark</u> $\mathcal{A}$ is AS-regular (Artin-Schelter) if $\mathcal{A}$ is regular and has polynomial growth i.e. dim$(\mathcal{A}_n) \leq Cn^{D'-1}$, GK-dim$(\mathcal{A}) = \text{Inf}(D')$.

PROPOSITION 1 $\mathcal{A}$ *regular* gldim$(\mathcal{A}) = D$.
(i) $D = 2 \Rightarrow \mathcal{A}$ *is quadratic and Koszul.*
(ii) $D = 3 \Rightarrow \mathcal{A}$ *is* N*-homogeneous with* $N \geq 2$ *and Koszul.*

For $D = 4$ there are examples of regular algebras which are not homogeneous.

$\boxed{D = 2}$

THEOREM 1 *$B = (B_{\mu\nu})$ nondegenerate bilinear form on $\mathbb{K}^d$, $\mathcal{A}$ quadratic generated by x^λ ($\lambda \in \{1, \dots, d\}$) with relation $B_{\mu\nu}x^\mu x^\nu = 0$. Then $\mathcal{A}$ is regular with* gldim$(\mathcal{A}) = 2$. *Conversely any $\mathcal{A}$ regular with* gldim$(\mathcal{A}) = 2$ *is of this type. $\mathcal{A} \simeq \mathcal{A}' \Leftrightarrow B$ and B' in the same GL$(d, \mathbb{K})$-orbit.*

Resolution $0 \rightarrow \mathcal{A} \stackrel{x^tB}{\rightarrow} \mathcal{A}^d \stackrel{x}{\rightarrow} \mathcal{A} \rightarrow \mathbb{K} \rightarrow 0 \Rightarrow$ Gorenstein by transpose using invertib. of $B_{\mu\nu}$.

$d = 2 \Rightarrow$ polyn. growth, in this case 3 types :

$$rk = 0 \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad x^1x^2 - x^2x^1 = 0$$

$$rk = 1 \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad x^1x^2 - x^2x^1 - (x^2)^2 = 0$$

$$rk = 2 \quad B = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix} \quad q \neq 0, q \neq 1 \text{ and } q \sim q^{-1}$$

$$x^1x^2 - qx^2x^1 = 0$$

$\boxed{D=2}$, *continuation*

$P_{\mathcal{A}}(t) = \sum_n \dim(\mathcal{A}_n) t^n = \frac{1}{1-dt+t^2}$

$\mathcal{H}$ generated by $T^\mu_\nu (\mu, \nu \in \{1, \dots, , d\})$ with relations

$B_{\mu\nu} T^\mu_\rho T^\nu_\sigma = B_{\rho\sigma} \mathbb{1}$ and $T^\mu_\rho T^\nu_\sigma B^{\rho\sigma} = B^{\mu\nu} \mathbb{1}$

is a Hopf algebra with

$\Delta T^\mu_\nu = T^\mu_\lambda \otimes T^\lambda_\nu$, $\varepsilon(T^\mu_\nu) = \delta^\mu_\nu$, $S(T^\mu_\nu) = B^{\mu\sigma} T^\rho_\sigma B_{\rho\nu}$

$\mathcal{H}$ corresponds to the *quantum group of the nondegenerate bilinear form* B while $\mathcal{A}$ corresponds to the natural quantum space for the action of this quantum group.

Case $d = 2 \Rightarrow$ usual, etc.

R-Matrices for $D = 2$

$$x^\mu x^\nu = R^{\mu\nu}_{\lambda\rho} x^\lambda x^\rho, \;\; R^{\mu\nu}_{\lambda\rho} = \delta^\mu_\lambda \delta^\nu_\rho + K^{\mu\nu} B_{\lambda\rho}$$

$$R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23}$$

$$\Leftrightarrow \begin{cases} KB\ {}^tK\ {}^tB + (1 + \mathrm{tr}(K\ {}^tB))\mathbb{1} = 0 \\ {}^tK\ {}^tBKB + (1 + \mathrm{tr}(K\ {}^tB))\mathbb{1} = 0 \end{cases}$$

$$\Rightarrow (R - \mathbb{1})(R - (1 + \mathrm{tr}(K\ {}^tB))\mathbb{1}) = 0$$

Case $d = 2$ with $B = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}$

$K = \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} \Rightarrow (R - \mathbb{1})(R + pq\mathbb{1}) = 0$

$p = q \Rightarrow K = qB^{-1} \Rightarrow (R - \mathbb{1})(R + q^2\mathbb{1}) = 0$

$p = q^{-1} \Rightarrow R^2 = \mathbb{1}$ $\quad$ ($\sim$ permutations)

Preregular Multilinear Forms

$n \in \mathbb{N}$ with $n \geq 1$
$W : V^{\otimes^{n+1}} \to \mathbb{K}$ $(n+1)$-linear form on V
W is *preregular* iff one has (i) and (ii) :
(i) $W(X, X_1, \dots, X_n) = 0 \ \forall X_p \in V \Rightarrow X = 0$,
(ii) $\exists Q_W \in \mathrm{GL}(V)$ such that $\forall X_\alpha \in V$
$W(X_0, \dots, X_{n-1}, X_n) = W(Q_W X_n, X_0, \dots, X_{n-1})$.
($\Rightarrow Q_W$ unique for given W).

One has then also
$W(X_1, \dots, X_p, X, X_{p+1}, \dots, X_n) = 0 \ \forall X_r \in V$
$\Rightarrow X = 0$.
(A bilinear form is preregular iff it is nondegenerate).

Action of $\mathrm{GL}(V)$
$W^L = W \circ L^{-1}$, $L \in \mathrm{GL}(V)$
$\Rightarrow Q_{W^L} = L Q_W L^{-1}$

One has $W^{Q_W} = W$

3-regular Multilinear Forms

$N \in \mathbb{N}$ with $N \geq 2$
W a $(N+1)$-linear form on V
W is 3-*regular* iff it is preregular and satisfies condition (iii) :
(iii) If $L_0, L_1 \in \mathrm{End}(V)$ are such that
$W(L_0X_0, X_1, X_2,.., X_N) = W(X_0, L_1X_1, X_2,.., X_N)$
$\forall X_\alpha \in V$ then $L_0 = L_1 = \lambda \mathbb{1}$ with $\lambda \in \mathbb{K}$.

Remark. The following condition $(iiii)$ is strictly stronger than (iii) :
$(iiii)$ If $\sum_i Y_i \otimes Z_i \in V \otimes V$ is such that
$\sum_i W(Y_i, Z_i, X_1, \dots, X_{N-1}) = 0$
$\forall X_p \in V$ then $\sum_i Y_i \otimes Z_i = 0$.

Example. Let $V = \mathbb{K}^{N+1}$, $\varepsilon(N+1) = (\varepsilon_{\lambda_0 \dots \lambda_N})$ compl. antisym. with $\varepsilon_{0\dots N} = 1$.
Then $\varepsilon(N+1)$ is 3-regular but $(iiii)$ is not satisfied.

$\boxed{D = 3}$

CONJECTURE 1 *Let* $W = (W_{\lambda_0\dots\lambda_N})$ *a 3-regular* $(N+1)$*-linear form on* $\mathbb{K}^d$*,* $\mathcal{A}$ *the* N*-homogeneous algebra generated by* x^λ *(*$\lambda \in \{1,\dots,d\}$*) with relations* $W_{\lambda\lambda_1\dots\lambda_N} x^{\lambda_1}\dots x^{\lambda_N} = 0$ $(\lambda \in \{1,\dots,d\})$*. Then* $\mathcal{A}$ *is regular with gldim*$(\mathcal{A}) = 3$*.*

THEOREM 2 *Any* $\mathcal{A}$ *regular with gldim*$(\mathcal{A}) = 3$ *is of the above type for some* d*, some* N *and some 3-regular* $(N+1)$*-linear form* W *on* $\mathbb{K}^d$*.* $\mathcal{A} \simeq \mathcal{A}' \Leftrightarrow W$ *and* W' *in the same GL*$(d,\mathbb{K})$*-orbit.*

Setting $M_{\lambda\mu} = W_{\lambda\nu_1\dots\nu_{N-1}\mu} x^{\nu_1} \otimes \dots \otimes x^{\nu_{N-1}}$
one has the resolution

$$0 \to \mathcal{A} \xrightarrow{x^t} \mathcal{A}^d \xrightarrow{M} \mathcal{A}^d \xrightarrow{x} \mathcal{A} \to \mathbb{K} \to 0$$

Homogeneous Algebras

$\mathbb{K}$ field of characteristic zero
$N \in \mathbb{N}$ with $N \geq 2$

$\mathcal{A}$ *N-homogeneous algebra* :
$\mathcal{A} = A(E, R) = T(E)/(R)$
$\dim(E) < \infty$, $R \subset E^{\otimes^N}$

$\Rightarrow \mathcal{A}$ connected graded algebra ($\mathcal{A}_0 = \mathbb{K}\mathbb{1}$)
generated in degree 1 ($\mathcal{A}_1 = E$).

$f : A(E, R) \to A(E', R')$ *morphism* :
$f \in \mathrm{Hom}_{\mathbb{K}}(E, E')$ such that $f^{\otimes^N}(R) \subset R'$
$\Rightarrow f$ induces an algebra homomorphism.

Category $\mathbf{H_N Alg}$
Forgetful functor $\mathcal{A} \mapsto E$ from $\mathbf{H_N Alg}$ to $\mathbf{Vect}$

Duality

$\mathcal{A} = A(E,R)$ N-homogeneous algebra
$\mapsto \mathcal{A}^! = A(E^*, R^\perp)$ *dual N-homogeneous algebra*

where

$R^\perp = \{\omega \in (E^{\otimes^N})^* | \omega(x) = 0,\ \forall x \in R\}$
with the identification $(E^{\otimes^N})^* = E^{*\otimes^N}$

$(\mathcal{A}^!)^! = \mathcal{A}$
$f : \mathcal{A} \to \mathcal{A}'$ morphism
$\mapsto f^! : (\mathcal{A}')^! \to \mathcal{A}^!$ morphism

$(\mathcal{A} \mapsto \mathcal{A}^!, f \mapsto f^!)$ involutive contravariant functor
$\mathcal{A} \mapsto \mathcal{A}^!$ is a lifting to $\mathbf{H_N Alg}$ of the duality $E \mapsto E^*$ in $\mathbf{Vect}$

Koszul N-complex

$\mathcal{A} = A(E, R)$
$K(\mathcal{A})$=chain N-complex of free left $\mathcal{A}$-modules

$K_n(\mathcal{A}) = \mathcal{A} \otimes \mathcal{A}_n^{!*}$

$\mathcal{A}_n^{!*} = E^{\otimes^n}$ if $n < N$

$\mathcal{A}_n^{!*} = \cap_{r+s+N=n} E^{\otimes^r} \otimes R \otimes E^{\otimes^s}$ if $n \geq N$

$\Rightarrow \mathcal{A}_n^{!*} \subset E^{\otimes^n}$, $\forall n \in \mathbb{N}$

$d : K_{n+1}(\mathcal{A}) \to K_n(\mathcal{A})$ induced by

$d(a \otimes (x_0 \otimes x_1 \otimes \cdots \otimes x_n)) = ax_0 \otimes (x_1 \otimes \cdots \otimes x_n)$

$\Rightarrow d^N = 0$ since $\mathcal{A}_n^{!*} \subset R \otimes E^{\otimes n-N}$ for $n \geq N$

Koszul Homogeneous Algebras

$$C_{p,r}, \quad 0 \leq r \leq N-2, \;\; r+1 \leq p \leq N-1$$

complexes obtained by *contraction* of $K(\mathcal{A})$

$$\dots \stackrel{d^{N-p}}{\rightarrow} \mathcal{A} \otimes \mathcal{A}^{!*}_{nN+r} \stackrel{d^{p}}{\rightarrow} \mathcal{A} \otimes \mathcal{A}^{!*}_{nN-p+r} \stackrel{d^{N-p}}{\rightarrow} \dots$$

$$\dots \stackrel{d^{p}}{\rightarrow} \mathcal{A} \otimes \mathcal{A}^{!*}_{N-p+r} \stackrel{d^{N-p}}{\rightarrow} \mathcal{A} \otimes \mathcal{A}^{!*}_{r} \rightarrow 0$$

PROPOSITION 2 $N \geq 3$, $(p,r) \neq (N-1,0)$

$$H_1(C_{p,r}) = 0 \Rightarrow R = 0 \textit{ or } R = E^{\otimes^N}.$$

$\mathcal{A}$ *Koszul N-homogeneous algebra* :

$$H_n(C_{N-1,0}) = 0, \quad \forall n \geq 1.$$

$\Rightarrow$ resolution of the trivial $\mathcal{A}$-module $\mathbb{K}$. $C_{N-1,0}$ will be denoted by $\mathcal{K}(\mathcal{A},\mathbb{K})$, it coincides with the *Koszul complex* introduced by Roland Berger.

Complex $\mathcal{K}(\mathcal{A},\mathcal{A})$

$K(\mathcal{A})$ N-complex of left $\mathcal{A}$-modules

$$\dots \xrightarrow{d} \mathcal{A}\otimes\mathcal{A}^{!*}_{n+1} \xrightarrow{d} \mathcal{A}\otimes\mathcal{A}^{!*}_{n} \xrightarrow{d} \dots$$

d induced by

$$a\otimes(e_1\otimes\cdots\otimes e_{n+1}) \mapsto ae_1\otimes(e_2\otimes\cdots\otimes e_{n+1})$$

$\tilde{K}(\mathcal{A})$ N-complex of right $\mathcal{A}$-modules

$$\dots \xrightarrow{\tilde{d}} \mathcal{A}^{!*}_{n+1}\otimes\mathcal{A} \xrightarrow{\tilde{d}} \mathcal{A}^{!*}_{n}\otimes\mathcal{A} \xrightarrow{\tilde{d}} \dots$$

$\tilde{d}$ induced by

$$(e_1\otimes\cdots\otimes e_{n+1})\otimes a \mapsto (e_1\otimes\cdots\otimes e_n)\otimes e_{n+1}a$$

$\Rightarrow$ two N-complexes of bimodules (L,R)

$$\dots \xrightarrow{d_L,d_R} \mathcal{A}\otimes\mathcal{A}^{!*}_{n+1}\otimes\mathcal{A} \xrightarrow{d_L,d_R} \mathcal{A}\otimes\mathcal{A}^{!*}_{n}\otimes\mathcal{A} \xrightarrow{d_L,d_R} \dots$$

$$d_L = d\otimes I_{\mathcal{A}}, \qquad d_R = I_{\mathcal{A}}\otimes\tilde{d}$$

...

$\boxed{\text{Complex } \mathcal{K}(\mathcal{A},\mathcal{A})}$, *continuation*

$$d_L d_R = d_R d_L \Rightarrow (d_L - d_R)\left(\sum_{p=0}^{N-1} d_L^p d_R^{N-p-1}\right) =$$

$$\left(\sum_{p=0}^{N-1} d_L^p d_R^{N-p-1}\right)(d_L - d_R) = d_L^N - d_R^N = 0$$

Define the chain complex of $(\mathcal{A},\mathcal{A})$-bimodules $\mathcal{K}(\mathcal{A},\mathcal{A})$ by

$$\mathcal{K}_{2m}(\mathcal{A},\mathcal{A}) = \mathcal{A} \otimes \mathcal{A}^{!*}_{Nm} \otimes \mathcal{A} = \mathcal{K}_{2m}(\mathcal{A},\mathbb{K}) \otimes \mathcal{A}$$

$$\mathcal{K}_{2m+1}(\mathcal{A},\mathcal{A}) = \mathcal{A} \otimes \mathcal{A}^{!*}_{Nm+1} \otimes \mathcal{A} = \mathcal{K}_{2m+1}(\mathcal{A},\mathbb{K}) \otimes \mathcal{A}$$

with differential δ' defined by

$$\delta' = d_L - d_R : \mathcal{K}_{2m+1}(\mathcal{A},\mathcal{A}) \to \mathcal{K}_{2m}(\mathcal{A},\mathcal{A})$$

$$\delta' = \sum_{p=0}^{N-1} d_L^p d_R^{N-p-1} : \mathcal{K}_{2(m+1)}(\mathcal{A},\mathcal{A}) \to \mathcal{K}_{2m+1}(\mathcal{A},\mathcal{A})$$

Properties of Koszul algebras

PROPOSITION 3 *$\mathcal{A} = A(E,R)$ N-homogeneous $H_n(\mathcal{K}(\mathcal{A},\mathcal{A})) = 0$ for $n \geq 1 \Leftrightarrow \mathcal{A}$ is Koszul.*

- $\mathcal{A}$ Koszul $\Leftrightarrow$ $\mathcal{K}(\mathcal{A},\mathbb{K}) \to \mathbb{K} \to 0$ is a (free) resolution of the trivial left $\mathcal{A}$-module $\mathbb{K}$
- $\mathcal{A}$ Koszul $\Leftrightarrow$ $\mathcal{K}(\mathcal{A},\mathcal{A}) \to \mathcal{A} \to 0$ is a (free) resolution of the $(\mathcal{A},\mathcal{A})$-bimodule $\mathcal{A}$.

$$P_{\mathcal{A}}(t) = \sum_n \mathrm{dim}(\mathcal{A}_n)t^n$$

$$Q_{\mathcal{A}}(t) = \sum_p (\mathrm{dim}(\mathcal{A}^!_{Np})t^{Np} - \mathrm{dim}(\mathcal{A}^!_{Np+1})t^{Np+1})$$

$\mathcal{A}$ Koszul $\Rightarrow Q_{\mathcal{A}}(t)P_{\mathcal{A}}(t) = 1$.
which generalizes a well-known result for quadratic algebras since in the latter case ($N = 2$) $Q_{\mathcal{A}}(t) = P_{\mathcal{A}^!}(-t)$.

Small complexes $\mathcal{S}(\mathcal{A},\mathcal{M})$

- $\mathcal{A} = A(E,R)$ N-homogeneous
- $\mathcal{M} = (\mathcal{A},\mathcal{A})$-bimodule
- $\mathcal{S}(\mathcal{A},\mathcal{M}) = \mathcal{M}\otimes_{\mathcal{A}\otimes\mathcal{A}^{\text{opp}}}\mathcal{K}(\mathcal{A},\mathcal{A})$ *small complex*

- If $\mathcal{A}$ is Koszul then the free resolution $\mathcal{K}(\mathcal{A},\mathcal{A}) \to \mathcal{A} \to 0$ of $\mathcal{A}\otimes\mathcal{A}^{\text{opp}}$-modules and the interpretation of the Hochschild homology as

$$H_n(\mathcal{A},\mathcal{M}) = \text{Tor}_n^{\mathcal{A}\otimes\mathcal{A}^{\text{opp}}}(\mathcal{M},\mathcal{A})$$

imply that the small complex $\mathcal{S}(\mathcal{A},\mathcal{M})$ computes the Hochschild homology i.e.

$$H_n(\mathcal{A},\mathcal{M}) = H_n(\mathcal{S}(\mathcal{A},\mathcal{M}))$$

for $n \in \mathbb{N}$.

Dimensions

- $\mathcal{A} = \oplus_{n\in\mathbb{N}}\mathcal{A}_n$ graded algebra
$\mathcal{A}$ has *polynomial growth* if

$$\dim_{\mathbb{K}}(\mathcal{A}_n) \leq Cn^{D-1}, \qquad \forall n \geq 1$$

GK-dim($\mathcal{A}$)= smallest D as above.

- $\mathcal{A}$ N-homogeneous and Koszul
then the above resolutions are minimal projective $\Rightarrow$
global dimension of $\mathcal{A}$ = smallest D
such that $\mathcal{K}_D(\mathcal{A},\mathbb{K}) \neq 0$ and
$\mathcal{K}_n(\mathcal{A},\mathbb{K}) = 0$ for $n > D$.
$\mathcal{A}$ has finite global dimension if
$\mathcal{A}^!_n = 0$ for $n >$ some integer.
Hochschild dim($\mathcal{A}$)= gldim($\mathcal{A}$)
Generically GK-dim($\mathcal{A}$) $\neq$ gldim($\mathcal{A}$).

Gorenstein Homogeneous Algebras

$\mathcal{L}(\mathcal{A},\mathbb{K})$ = dual of $\mathcal{K}(\mathcal{A},\mathbb{K})$
$\mathcal{L}(\mathcal{A},\mathbb{K})$ is a cochain complex of right $\mathcal{A}$-modules (finite, free) and

$$\mathcal{L}(\mathcal{A},\mathbb{K}) = C_{1,0}(L(\mathcal{A}))$$

where $L(\mathcal{A})$ is the cochain N-complex of right $\mathcal{A}$-modules "dual of $K(\mathcal{A})$".

$\mathcal{A}$ N-homogeneous and Koszul of finite global dimension $D \Rightarrow \mathcal{L}^n(\mathcal{A},\mathbb{K}) = 0$ for $n > D$.
Then $\mathcal{A}$ is *Gorenstein* if $\mathcal{L}(\mathcal{A},\mathbb{K})$ gives a (minimal projective) resolution

$$0 \to \mathcal{L}^0(\mathcal{A},\mathbb{K}) \xrightarrow{d} \dots \xrightarrow{d} \mathcal{L}^D(\mathcal{A},\mathbb{K}) \to \mathbb{K} \to 0$$

of the trivial right $\mathcal{A}$-module $\mathbb{K}$.
This implies

$$\mathcal{K}_n(\mathcal{A},\mathbb{K}) \simeq \mathcal{K}_{D-n}(\mathcal{A},\mathbb{K})$$

Multinear Forms and Homogeneous Algebras

$m \geq N \geq 2$, $W = m$-linear form on $\mathbb{K}^d$

$\mathcal{A} = \mathcal{A}(W, N)$ generated by $x^\lambda \in \{1, \dots, d\})$ with relations

$W_{\lambda_1 \dots \lambda_{m-N} \mu_1 \dots \mu_N} x^{\mu_1} \dots x^{\mu_N} = 0$ $(\lambda_i \in \{1, \dots, d\})$

i.e. $\mathcal{A} = A(E, R)$, $E = \oplus_\lambda \mathbb{K} x^\lambda$ and

$R = \sum_{\lambda_i} \mathbb{K} W_{\lambda_1 \dots \lambda_{m-N} \mu_1 \dots \mu_N} x^{\mu_1} \otimes \dots \otimes x^{\mu_N} \subset E^{\otimes^N}$

$\underline{W}^{(n)} = \sum_{\lambda_i} \mathbb{K} W_{\lambda_1 \dots \lambda_n \mu_1 \dots \mu_{m-n}} x^{\mu_1} \otimes \dots \otimes x^{\mu_{m-n}} \subset E^{\otimes^{m-n}}$

$$\mathcal{W}_n = \begin{cases} \mathcal{A} \otimes \underline{W}^{(m-n)} & \text{if } m \geq n \geq N \\ \mathcal{A} \otimes E^n & \text{if } N > n \end{cases}$$

PROPOSITION 4 *Assume W preregular. The sequence*
$0 \to \mathcal{W}_m \xrightarrow{d} \mathcal{W}_{m-1} \xrightarrow{d} \dots \xrightarrow{d} \mathcal{W}_0 \to 0$
is a sub-N-complex of $K(\mathcal{A})$.

Pre Frobenius Structure of $\mathcal{A}^!$

One has $W \in \cap_r E^{\otimes^{m-N-r}} \otimes R \otimes E^{\otimes^r} = \mathcal{A}^{!*}_m$

W composed with $\mathcal{A}^! \to \mathcal{A}^!_m$ is a linear form ω_W on $\mathcal{A}^!$.

PROPOSITION 5 *Let W be as in last proposition.*
(i) $Q_W \in \mathrm{GL}(d,\mathbb{K}) = \mathrm{GL}(E^*)$ induces an automorphism σ_W of $\mathcal{A}^!$.
(ii) One has $\omega_W(xy) = \omega_W(\sigma_W(y)x)$, $\forall x, y \in \mathcal{A}^!$

i.e. $\mathcal{A}^!$ is pre Frobenius and

$I_W = \{x \in \mathcal{A}^! | \omega_W(xy) = 0,\ \forall y \in \mathcal{A}^!\}$

is a two-sided ideal $\Rightarrow$

$\mathcal{A}^!/I_W$ is a Frobenius algebra.

The Canonical Volume $\mathbb{1} \otimes W$

$W = D$-linear preregular $\Rightarrow Q_W$
$\mathcal{A} = \mathcal{A}(W, 2)$ quadratic algebra

LEMMA 1 *There is a unique* $\tilde{\sigma}_W \in$ *Aut*$(\mathcal{A})$ *such that* $\tilde{\sigma}_W(Q^\alpha_{W\beta} x^\beta) = (-1)^{D-1} x^\alpha$

$\tilde{\sigma}_W$ is graded $\tilde{\sigma}_W(\mathcal{A}_n) \subset \mathcal{A}_n$

${}^W\!\mathcal{A}$=bimodule defined by :
${}^W\!\mathcal{A} = \mathcal{A}$ as a right-module with left multiplication by a given by $\varphi \mapsto \tilde{\sigma}_W(a)\varphi$.

THEOREM 3

$$\mathbb{1} \otimes W \in Z_D(\mathcal{A}, {}^W\!\mathcal{A}).$$

Koszul Regular Algebras

THEOREM 4 *$\mathcal{A}$ N-homogeneous, Koszul-Gorenstein with gldim($\mathcal{A}$) $= D < \infty$ generated by d elements. Then $\mathcal{A} = \mathcal{A}(W, N)$ for some preregular m-linear form W on $\mathbb{K}^d$. If $N \geq 3$ one has $m = Np + 1$ and $D = 2p + 1$ for some integer $p \geq 1$ while for $N = 2$ one has $m = D$.*

$$\nu_N(2p) = Np, \quad \nu_N(2p+1) = Np + 1 \quad (p \in \mathbb{N})$$

$$E(\mathcal{A}) = \oplus_n E(\mathcal{A})_n \quad \text{with} \quad E(\mathcal{A})_n = \mathcal{A}^!_{\nu_N(n)}$$

$x * y = (-1)^{ij} xy$ if $N = 2$ or if $N > 2$ and i or j even, $x * y = 0$ if $N > 2$ and i and j odd.

$\Rightarrow E(\mathcal{A})$ is a graded algebra $\simeq \mathrm{Ext}^\bullet_{\mathcal{A}}(\mathbb{K}, \mathbb{K})$ (= Yoneda).

THEOREM 5 *$\mathcal{A}$ Koszul N-homogeneous with gldim($\mathcal{A}$) $= D < \infty$. Then $\mathcal{A}$ is Gorenstein iff $E(\mathcal{A})$ is Frobenius.*

Yang-Mills Algebra

Yang-Mills algebra = cubic algebra $\mathcal{A}$ generated by ∇_λ, $\lambda \in \{0,\dots,s\}$ with relations

$$g^{\lambda\mu}[\nabla_\lambda,[\nabla_\mu,\nabla_\nu]] = 0,\ \nu \in \{0,\dots,s\}$$

$\Rightarrow \mathcal{A} = U(\mathfrak{g})$, $\mathfrak{g} = \sum_{k\geq}\mathfrak{g}_k$ graded Lie algebra

THEOREM 6 *$\mathcal{A}$ is Koszul of global dim $= 3$ and is Gorenstein.*

$\mathcal{A}^!$ generated by θ^λ (dual basis of ∇_λ)

$$\theta^\lambda\theta^\mu\theta^\nu = \frac{1}{s}(g^{\lambda\mu}\theta^\nu + g^{\mu\nu}\theta^\lambda - 2g^{\lambda\nu}\theta^\mu)\mathbf{g}$$

where $\mathbf{g} = g_{\alpha\beta}\theta^\alpha\theta^\beta \in \mathcal{A}^!_2$
$\Rightarrow \mathbf{g}$ central.

$$W^{\alpha_0\alpha_1\alpha_2\alpha_3} = g^{\alpha_1\alpha_2}g^{\alpha_3\alpha_0} + g^{\alpha_2\alpha_3}g^{\alpha_1\alpha_0} - 2g^{\alpha_1\alpha_3}g^{\alpha_2\alpha_0}$$

$$\Rightarrow Q_W = \mathbb{1}$$

W satisfies $(iiii) \Rightarrow \mathcal{A}^!$ is Frobenius.

Super Yang-Mills Algebra

Super Lie algebra version = cubic algebra $\tilde{\mathcal{A}}$ generated by S_λ, $\lambda \in \{0,\dots,s\}$ with relations

$$g^{\lambda\mu}[S_\lambda, \{S_\mu, S_\nu\}] = 0,\ \nu \in \{0,\dots,s\}$$

equivalently

$$[g^{\lambda\mu} S_\lambda S_\mu, S_\nu] = 0, \quad \nu \in \{0,\dots,s\}$$

THEOREM 7 *$\tilde{\mathcal{A}}$ is Koszul of global dim = 3 and is Gorenstein.*

$\tilde{\mathcal{A}}^!$ generated by θ^λ (dual basis of S_λ)

$$\theta^\lambda\theta^\mu\theta^\nu = -\frac{1}{s}(g^{\lambda\mu}\theta^\nu - g^{\mu\nu}\theta^\lambda)\mathbf{g}$$

where $\mathbf{g} = g_{\alpha\beta}\theta^\alpha\theta^\beta \in \tilde{\mathcal{A}}^!_2$
$\Rightarrow \mathbf{g}\theta^\lambda + \theta^\lambda\mathbf{g} = 0$.

$W^{\alpha_0\alpha_1\alpha_2\alpha_3} = g^{\alpha_1\alpha_2}g^{\alpha_3\alpha_0} - g^{\alpha_2\alpha_3}g^{\alpha_1\alpha_0} \Rightarrow Q_W = -\mathbb{1}$

W satisfies $(iiii) \Rightarrow \tilde{\mathcal{A}}^!$ is Frobenius.

Calabi-Yau Algebras

$\mathcal{A}\otimes\mathcal{A}$	2 bimod. structures
outer	$a(x\otimes y)b = ax\otimes yb$
inner	$a(x\otimes y)b = xb\otimes ay$

$$\Rightarrow \text{Hom}_{\mathcal{A}-\mathcal{A}}(\mathcal{M}, \mathcal{A}\otimes\mathcal{A}),\ H^k(\mathcal{A}, \mathcal{A}\otimes\mathcal{A})$$

bimodules " for the inner struct."

$\mathcal{A}$ is *Calabi-Yau of dimension* n

$$\text{if } H^k(\mathcal{A}, \mathcal{A}\otimes\mathcal{A}) = \begin{cases} \mathcal{A} & \text{for } k = n \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ Hochschild-dim$(\mathcal{A}) = n$
(Proj.dim $(\mathcal{A})$ comme $\mathcal{A}\otimes\mathcal{A}^0$-mod)

THEOREM 8 *Every connected graded 3-dimensional Calabi-Yau algebra is of the form* $\mathcal{A} = A(W, N)$ *for some 3-regular* $(N+1)$*-linear form* W *with* $Q_W = \mathbb{1}$ *(i.e.* W *cyclic)*

Precommutative Examples

$m = d \geq N \geq 2$

$(W_{\lambda_1 \dots \lambda_d}) = (\varepsilon_{\lambda_1 \dots \lambda_d}) = \varepsilon(d)$

THEOREM 9 *$\mathcal{A} = \mathcal{A}(\varepsilon(d), N)$ is Koszul with gldim($\mathcal{A}$) $= D < \infty$. $\mathcal{A}$ is Gorenstein iff $N = 2$ or $d = Nq + 1 (= m)$ for some $q \in \mathbb{N}$.*
If $d = Nq + 1$ then $D = 2q + 1$.

For instance $D = 3$ for $d = N + 1$. In this case $I_{\varepsilon(N+1)} \neq 0$ and

$$\mathcal{A}^!/I_{\varepsilon(N+1)} \simeq \wedge\mathbb{K}^{N+1} \simeq \mathcal{A}(\varepsilon(N+1), 2)^!$$

$$\mathcal{A}(\varepsilon(N+1), 2) \simeq \mathbb{K}[X_0, \dots, X_N] \simeq S\mathbb{K}^{N+1}$$

$$\boxed{\mathcal{A}_{\mathbf{u}}}$$

$$\mathcal{A}_{\mathbf{u}} \left\{ \begin{array}{l} \cos(\varphi_0 - \varphi_k)[x^0, x^k] = i \sin(\varphi_\ell - \varphi_m)\{x^\ell, x^m\} \\ \cos(\varphi_\ell - \varphi_m)[x^\ell, x^m] = i \sin(\varphi_0 - \varphi_k)\{x^0, x^k\} \end{array} \right.$$

Koszul of global dim. $D = 4$ and Gorenstein if $\tilde{\mathrm{ch}}_{3/2}(U_{\mathbf{u}}) \neq 0$ and

$$\begin{aligned} W_{\lambda\mu\nu\rho} &= -\epsilon_{\lambda\mu\nu\rho} \cos(\varphi_\lambda - \varphi_\mu + \varphi_\nu - \varphi_\rho) \\ &+ i\delta_{\lambda\nu}\delta_{\mu\rho} \sin(\varphi_\lambda - \varphi_\mu + \varphi_\nu - \varphi_\rho) \end{aligned}$$

$$W_{\lambda\mu\nu\rho} = -W_{\rho\lambda\mu\nu}$$

$Q_W = -\mathbb{1}$, i.e. $\sigma_W = (-1)^{\text{degree}} \times$ Identity

Thus $W_{\lambda\mu\nu\rho} x^\lambda \otimes x^\mu \otimes x^\nu \otimes x^\rho = \tilde{\mathrm{ch}}_{\frac{3}{2}}(U_{\mathbf{u}})$ and the 2 natural volumes $\mathbb{1} \otimes W$ and $\mathbb{1} \otimes \tilde{\mathrm{ch}}_{3/2}$ coincide.

$\boxed{\mathcal{A}_{\mathbf{u}}}$, *complement*

$$W_{\lambda\mu\nu\rho}\ x^\nu \otimes x^\rho =$$

$$\cos(\varphi_\lambda - \varphi_\mu)\,\frac{1}{2}\sum_{\nu,\rho}\varepsilon_{\lambda\mu\nu\rho}R_{\nu\rho} - i\sin(\varphi_\lambda - \varphi_\mu)R_{\lambda\mu}$$

with

$$\begin{aligned} R_{\lambda\mu} &= \cos(\varphi_\lambda - \varphi_\mu)[x^\lambda, x^\mu]_\otimes \\ &\quad -\frac{i}{2}\sum_{\nu,\rho}\ \varepsilon_{\lambda\mu\nu\rho}\sin(\varphi_\nu - \varphi_\rho)\{x^\nu, x^\rho\}_\otimes \end{aligned}$$

formally

$$W^{(2)}_{\lambda\mu} = \cos(\varphi_\lambda - \varphi_\mu)(\star R)_{\lambda\mu} - i\sin(\varphi_\lambda - \varphi_\mu)R_{\lambda\mu}$$

3-Dimensional Sklyanin Algebra

$$\mathcal{A}\left\{\begin{array}{l} xy - qyx = pz^2 \\ yz - qzy = px^2 \\ zx - qxz = py^2 \end{array}\right.$$

$$\begin{aligned} W &= x\otimes y\otimes z + y\otimes z\otimes x + z\otimes x\otimes y \\ &- q(x\otimes z\otimes y + y\otimes x\otimes z + z\otimes y\otimes x) \\ &- p(x\otimes x\otimes x + y\otimes y\otimes y + z\otimes z\otimes z) \end{aligned}$$

i.e.

$$W_{ijk} = \tfrac{1}{2}(\varepsilon_{ijk} + |\varepsilon_{ijk}|) + \tfrac{q}{2}(\varepsilon_{ijk} - |\varepsilon_{ijk}|) - p\sum_{n=1}^{3}\delta_i^n\delta_j^n\delta_k^n$$

$$\Rightarrow Q_W = \mathbb{1}$$

$$\boxed{\mathcal{A} = C_{\mathbf{Alg}}(\mathbb{R}^d_{\mathbf{q}})}$$

$q^{\mu\nu} \in \mathbb{K}\,;\ \mu, \nu \in \{1, \dots, d\}$ such that
$q^{\mu\nu} q^{\nu\mu} = 1, \quad q^{\lambda\lambda} = 1$
$\mathcal{A} = C_{\mathbf{Alg}}(\mathbb{R}^d_{\mathbf{q}})$ defined by

$$x^\mu x^\nu = q^{\mu\nu} x^\nu x^\mu$$

$$S_d \hookrightarrow \left\{ \prod_{(\mu\nu)} b^{\mu\nu}, \ \ \mu < \nu \right\} \subset B_d$$

$$\chi : S_d \to \mathbb{K}, \quad \chi(\pi) = \prod_{(\mu\nu)} (-q^{\mu\nu})$$

$$W = \sum_{\pi \in S_d} \chi(\pi) x^{\pi(1)} \otimes \cdots \otimes x^{\pi(d)}$$

$$\Rightarrow (Q_W)^\mu_\nu = \left(\prod_{\lambda \neq \mu} (-q^{\lambda\mu}) \right) \delta^\mu_\nu$$

Infinitesimal Version

$W_t = \varepsilon + t\dot{W} + o(t^2)$
$Q_t = Q_{W_t} = (-1)^{d-1}\mathbb{1} + t\dot{Q} + o(t^2)$
Order 1 in t of Q-cyclicity :

$$\dot{W}_{\lambda_1 \ldots \lambda_d} = \dot{Q}^{\lambda}_{\lambda_d} \varepsilon_{\lambda\lambda_1 \ldots \lambda_{d-1}} + (-1)^{d-1} \dot{W}_{\lambda_d \lambda_1 \ldots \lambda_{d-1}}$$

$\Rightarrow \sum_\lambda \dot{Q}^\lambda_\lambda = 0 \Rightarrow \det(Q_t) = 1 + o(t^2)$

$\Rightarrow$ Natural question

$$det(Q_W) \overset{?}{=} 1$$

in the quadratic case with polynomial growth

NO in general, see next example.

Type E Quadratic Artin-Schelter Algebra

ξ = primitive 9-th root of 1

$$\mathcal{A}\begin{cases} x^2 + \xi^{-1}yz + \xi zy = 0 \\ y^2 + \xi^{-4}zx + \xi^4 xz = 0 \\ z^2 + \xi^{-7}xy + \xi^7 yx = 0 \end{cases}$$

$$\begin{aligned} W &= x\otimes z\otimes x + y\otimes x\otimes y + z\otimes y\otimes z \\ &+ \xi\ z\otimes x\otimes x + \xi^{-1}x\otimes x\otimes z \\ &+ \xi^4 x\otimes y\otimes y + \xi^{-4}y\otimes y\otimes x \\ &+ \xi^7 y\otimes z\otimes z + \xi^{-7}z\otimes z\otimes y \end{aligned}$$

$$\Rightarrow Q_W = \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi^4 & 0 \\ 0 & 0 & \xi^7 \end{pmatrix} = \xi \begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi^3 & 0 \\ 0 & 0 & \xi^6 \end{pmatrix}$$

$\Rightarrow$ tr $Q_W = 0$ et

det $Q_W = \xi^3 \neq 1$

Quantum Groups

$$m \geq N \geq 2$$

W m-linear on $\mathbb{K}^d$ pregular
$\exists$ $\tilde{W}$ m-linear on $\mathbb{K}^{d*}$ such that

$$\tilde{W}^{\mu\lambda_1\dots\lambda_{m-1}} W_{\lambda_1\dots\lambda_{m-1}\nu} = \delta^\mu_\nu$$

$\mathcal{H} = \mathcal{H}(W, \tilde{W})$ generated by u^α_β with

$$\begin{cases} W_{\alpha_1\dots\alpha_m} u^{\alpha_1}_{\beta_1} \dots u^{\alpha_m}_{\beta_m} = W_{\beta_1\dots\beta_m} \mathbb{1} \\ \tilde{W}_{\beta_1\dots\beta_m} u^{\alpha_1}_{\beta_1} \dots u^{\alpha_m}_{\beta_m} = W_{\alpha_1\dots\alpha_m} \mathbb{1} \end{cases}$$

$\mathcal{H}$ is a Hopf algebra with

$$\begin{cases} \Delta u^\alpha_\beta = u^\alpha_\gamma \otimes u^\gamma_\beta \\ \varepsilon(u^\alpha_\beta) = \delta^\alpha_\beta \\ S(u^\alpha_\beta) = \tilde{W}^{\alpha\lambda_1\dots\lambda_{m-1}} u^{\rho_1}_{\lambda_1} \dots u^{\rho_{m-1}}_{\lambda_{m-1}} W_{\rho_1\dots\rho_{m-1}\beta} \end{cases}$$

It coacts on $\mathcal{A}(W, N)$ via

$$\Delta_L(x^\mu) = u^\mu_\nu \otimes x^\nu$$

Quantum Groups, *continuation*

One has

$$\tilde{W}^{\lambda\rho_1\dots\rho_{m-1}}W_{\mu\rho_1\dots\rho_{m-1}} = \left(Q_W^{-1}\right)^\lambda_\mu$$

If $\tilde{W}$ is Q_W-invariant, there is a character χ on $\mathcal{H}$ such that

$$\chi(u^\alpha_\beta) = (Q_W)^\alpha_\beta$$

In the case $D = 2$, i.e. $W = B$
$\Rightarrow \tilde{W} = B^{-1}$ ($B=$ nondegenerate bilinear)

χ is a *sovereign* character on the Hopf algebra $\mathcal{H}$.

Around Conjecture 1 for $D = 3$

W preregular $(N+1)$-linear form on $\mathbb{K}^d$ identified with $W \in E^{\otimes^{N+1}}$, $\quad E = \mathbb{K}^{d*} = \oplus_\lambda \mathbb{K}x^\lambda$
$\mathcal{A} = \mathcal{A}(W, N)$ generated by $E = \mathcal{A}_1$

PROPOSITION 6 $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$
(i) W *is 3-regular*
(ii) $(\mathcal{A}^!_{N+1})^* = \mathbb{K}W$
(iii) *The Koszul complex* $\mathcal{K}(\mathcal{A}, \mathbb{K})$ *is the sequence*

$$0 \rightarrow \mathcal{A} \otimes W \stackrel{d}{\rightarrow} \mathcal{A} \otimes R \stackrel{d^{N-1}}{\rightarrow} \mathcal{A} \otimes E \stackrel{d}{\rightarrow} \mathcal{A} \rightarrow 0$$

COROLLARY 1 W *3-regular and* $\mathcal{A}$ *Koszul imply* $\mathcal{A}$ *regular with gldim*($\mathcal{A}$)*=3.*

It is the same here to assume $\mathcal{A}$ Koszul as to assume gldim($\mathcal{A}$)=3.

Filtered Koszul Algebras

$T(E)$ with filtration $F^n = \oplus_{p\leq n} E^{\otimes^p}$
$\pi_n : F^n \to E^{\otimes^n}$ ($E^{\otimes^n} \oplus F^{n-1} = F^n$)
$P \subset F^N$, $R = \pi_N(P) \subset E^{\otimes^N}$ subspaces
$I(P)$ filtered ideal $\Rightarrow$
$\mathcal{A} = T(E)/I(P)$ filtered algebra
Gr($\mathcal{A}$)=associated graded algebra

$p : A(E,R) \to$ Gr($\mathcal{A}$) surjective homomorphism of graded algebra induced by
$E^{\otimes^n} \to F^n \to F^n/(I(P)^n + F^{n-1})$

- $\mathcal{A}$ *has the* PBW *property* iff. p is an isomorphism

- $\mathcal{A}$ *is Koszul* iff. $\mathcal{A}$ has the PBW property and $A(E,R)$ is Koszul.

Criteria

THEOREM 10 *Assume that one has*
(i) $P \cap F^{N-1} = 0$
(ii) $(PE + EP) \cap F^N \subset P$
and that $A(E,R)$ *is Koszul. Then* $\mathcal{A}$ *is Koszul.*

In the following P satisfies $(i) \Rightarrow$
$P = \{x - \varphi(x) | x \in R\}$
$\varphi = \sum_0^{N-1} \varphi_p : R \to F^{N-1}$, $\varphi_p : R \to E^{\otimes p}$

PROPOSITION 7 *Let* P *satisfy* (i) *and* $\mathcal{W}_{N+1} = E \otimes R \cap R \otimes E$*. Then* (ii) *is equivalent to*

(1) $(I \otimes \varphi_{N-1} - \varphi_{N-1} \otimes I)(\mathcal{W}_{N+1}) \subset R$,

(2) $(\varphi_p(I \otimes \varphi_{N-1} - \varphi_{N-1} \otimes I) +$
$+ I \otimes \varphi_{p-1} - \varphi_{p-1} \otimes I)(\mathcal{W}_{N+1}) = 0$
for $1 \leq p \leq N-1$ *and*

(3) $\varphi_0(I \otimes \varphi_{N-1} - \varphi_{N-1} \otimes I)(\mathcal{W}_{N+1}) = 0$.

Examples

Yang-Mills inhomogeneous

Solving (1), (2), (3) of last proposition for Yang-Mills algebra gives Koszul algebras

$\varphi_k(W^\rho) = c^{\alpha_1 \dots \alpha_k \rho} \nabla_{\alpha_1} \otimes \cdots \otimes \nabla_{\alpha_k}$

$W^\rho = g^{\lambda\mu} g^{\nu\rho} [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]_\otimes]_\otimes$

$c^{\alpha\beta\gamma} = (g^{\alpha\rho} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\rho}) b_\rho + \omega^{[\alpha\beta\gamma]} + s^{(\alpha\beta\gamma)}$

with $s^{(\alpha\beta\rho)} b_\rho = 0$

$c^{\alpha\beta} = -\frac{1}{2} \omega^{[\alpha\beta\rho]} b_\rho + s^{(\alpha\beta)}$ with $s^{(\alpha\rho)} b_\rho = 0$

$c^\alpha = s^\alpha$

with $s^\rho b_\rho = 0$

Super Yang-Mills inhomogeneous

$c^{\alpha\beta\gamma} = g^{\alpha\gamma} b^\beta - g^{\beta\gamma} b^\alpha$

$c^{\alpha\beta} = \omega^{[\alpha\beta]}$

$c^\alpha = \frac{1}{2} \omega^{[\beta\alpha]} b_\beta$

Inhomogeneous $\mathcal{A}_{\mathbf{u}}$

$$\begin{aligned} R_{\lambda\mu} &= \cos(\varphi_\lambda - \lambda_\mu)[x^\lambda, x^\mu]_\otimes \\ &-\frac{i}{2}\sum_{\nu\rho} \varepsilon_{\lambda\mu\nu\rho} \sin(\varphi_\nu - \varphi_\rho)\{x^\nu, x^\rho\}_\otimes \end{aligned}$$

PBW - terms

$$\begin{aligned} \varphi(R_{\lambda\mu}) &= i\sum_{\nu,\rho} \varepsilon_{\lambda\mu\nu\rho}\ \cos(\varphi_\nu - \varphi_\rho)(V^\nu x^\rho - V^\rho x^\nu) \\ &+ i\sum_{\nu,\rho}\varepsilon_{\lambda\mu\nu\rho} \sin(\varphi_\nu - \varphi_\rho)\ V^\nu V^\rho \end{aligned}$$

wherer V^λ are scalars
Introduce a new generator x^5 with relations

$$[x^\lambda, x^5] = 0$$

and "homogenize"

$$\cos(\varphi_\lambda - \varphi_\mu)[x^\lambda, x^\mu] =$$

$$\frac{i}{2}\sum_{\nu,\rho} \varepsilon_{\lambda\mu\nu\rho}\{\sin(\varphi_\nu - \varphi_\rho)(\{x^\nu, x^\rho\} + 2V^\nu V^\rho (x^5)^2)$$

$$+2\cos(\varphi_\nu - \varphi_\rho)(V^\nu x^\rho - V^\rho x^\nu)x^5$$

$\Rightarrow$ 5-dim Koszul-Gorenstein, etc.