Noncommutative motives,

Thermodynamics and the zeros of zeta

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Work in progress

The BC system

$$P_R = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \; ; \; a, b \in R \, , \; a \text{ invertible} \right\}$$

$P_{\mathbb{Z}}^{+} \subset P_{\mathbb{Q}}^{+}$ almost normal

$$(f_1 * f_2)(\gamma) = \sum_{\Gamma_0 \backslash \Gamma} f_1(\gamma \gamma_1^{-1}) f_2(\gamma_1) \, .$$

$$\mathcal{H}_{\mathbb{C}}(\Gamma, \Gamma_0) = \mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0) \otimes_{\mathbb{Q}} \mathbb{C}$$

$$\sigma_t(f)(\gamma) = \left(\frac{L(\gamma)}{R(\gamma)} \right)^{-it} f(\gamma)$$

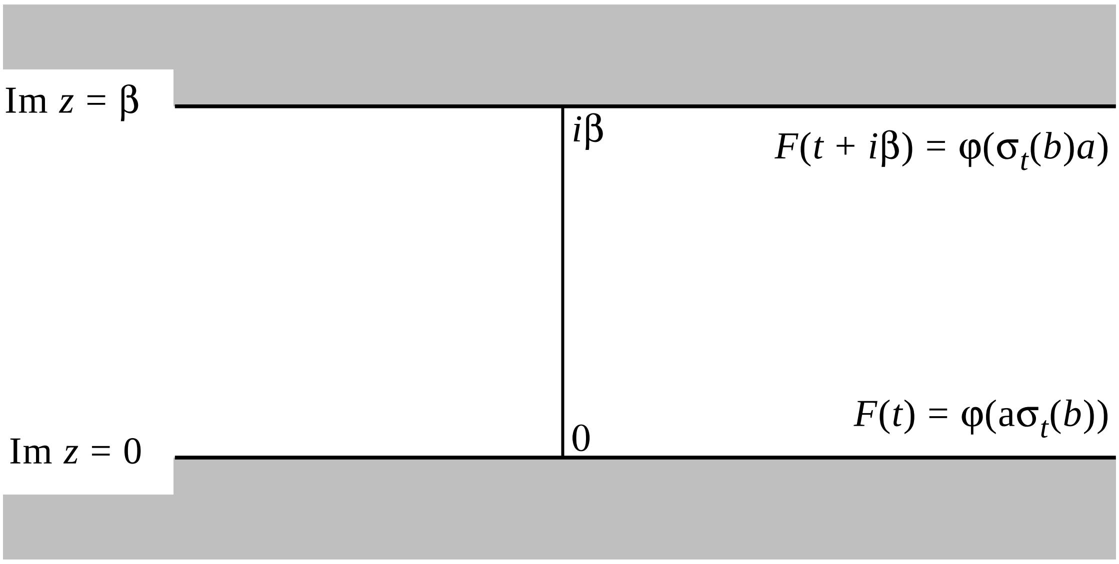
$L(\gamma) =$ Cardinality of left Γ_0 orbit of γ in Γ / Γ_0 ,

$$R(\gamma) = L(\gamma^{-1}) \, .$$

The KMS condition

$$\varphi(x^*x) \geq 0 \qquad \forall\, x \in \mathcal{A}\,,\ \varphi(1) = 1\,.$$

$$\sigma_t \in \mathrm{Aut}(\mathcal{A})$$



$$F_{x,y}(t) = \varphi(x\sigma_t(y))$$

$$F_{x,y}(t + i\beta) = \varphi(\sigma_t(y)x), \qquad \forall t \in \mathbb{R}.$$

Theorem (BC) The structure of KMS states is the following.

– $\mathcal{E}_\beta$ is a singleton for all $0 < \beta \leq 1$. This unique KMS state, restricted to the subalgebra $\mathcal{B}_{\mathbb{Q}}$ takes values

$$\varphi_\beta(e(a/b)) = f_{-\beta+1}(b)/f_1(b),$$

where

$$f_k(b) = \sum_{d|b} \mu(d)(b/d)^k,$$

with μ the Möbius function, and f_1 is the Euler totient function.

– Within the range of temperatures $1 < \beta < \infty$, the values of states $\varphi_{\beta,\rho} \in \mathcal{E}_\beta$ on the elements $e(r) \in \mathcal{B}_{\mathbb{Q}}$ are given by polylogarithm functions evaluated at roots of unity, normalized by the Riemann zeta function,

$$\varphi_{\beta,\rho}(e(r)) = \frac{1}{\zeta(\beta)} \mathrm{Li}_\beta(\rho(\zeta_r)),$$

where $\mathrm{Li}_s(z) = \sum_{n=1}^\infty z^n/n^s$.

– The group $\mathrm{GL}_1(\hat{\mathbb{Z}})$ acts by automorphisms of the system $(\mathcal{A}, \sigma_t)$. The induced action of $\mathrm{GL}_1(\hat{\mathbb{Z}})$ on $\mathcal{E}_\beta$ below critical temperature is free and transitive.

– The extreme KMS states $\mathcal{E}_\infty$ at zero temperature have the property that

$$\varphi(\mathcal{A}_{\mathbb{Q}}) \subset \mathbb{Q}^{cycl}, \quad \forall \varphi \in \mathcal{E}_\beta.$$

Moreover, the class field theory isomorphism intertwines the Galois action on values with the action of $\hat{\mathbb{Z}}^*$ by symmetries,

$$\gamma\, \varphi(f) = \varphi(\theta^{-1}(\gamma)\, f),$$

for all $\varphi \in \mathcal{E}_\infty$, for all $\gamma \in \mathrm{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q})$ and for all $f \in \mathcal{A}_{\mathbb{Q}}$.

Prime Numbers

$\pi(n) =$ number of prime numbers $p \leq n$

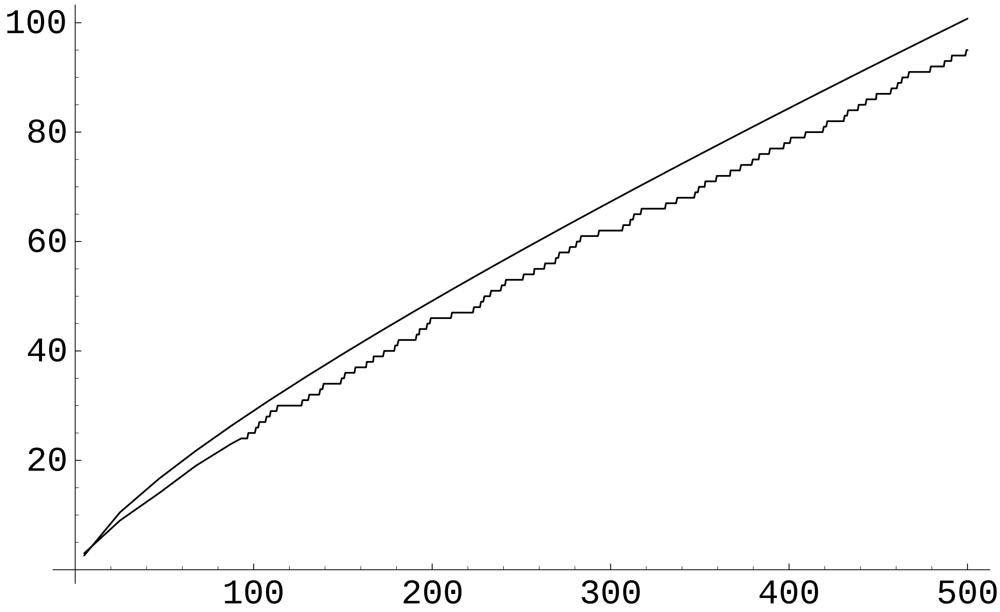
$$Li(x) = \int_0^x \frac{du}{\log(u)} \sim \sum (k-1)! \frac{x}{\log(x)^k}$$

$$\pi(x) = \int_0^x \frac{du}{\log(u)} + R(x)$$

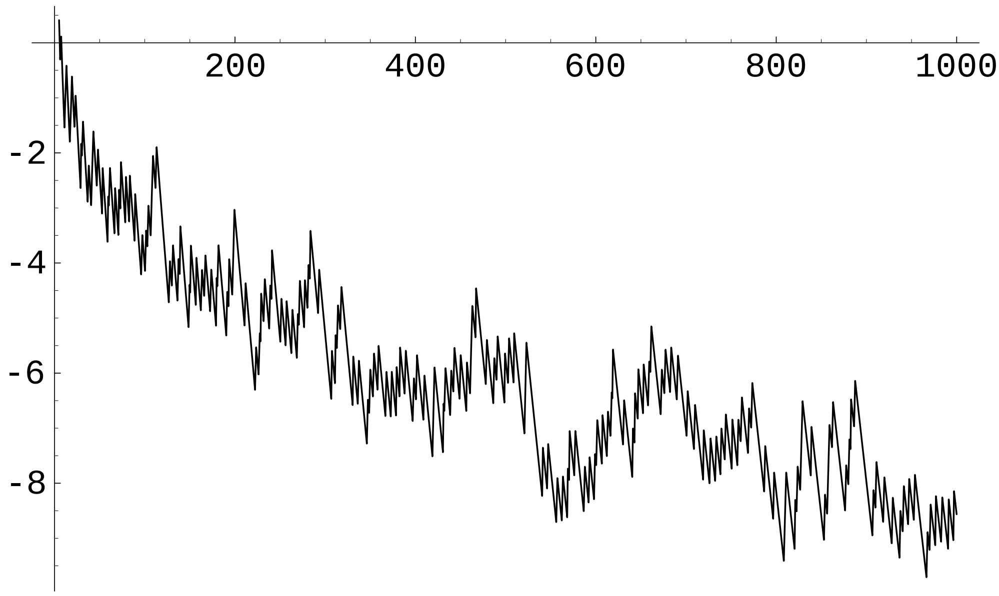
Riemann Conjecture :

$$R(x) = O(\sqrt{x} \log(x))$$

$$\left(\pi(n) = 2 + \sum_5^n \frac{e^{2\pi i \Gamma(k)/k} - 1}{e^{-2\pi i/k} - 1}, \quad \Gamma(k) = (k-1)!\right)$$



Graphs of $\pi(x)$ and Li(x)



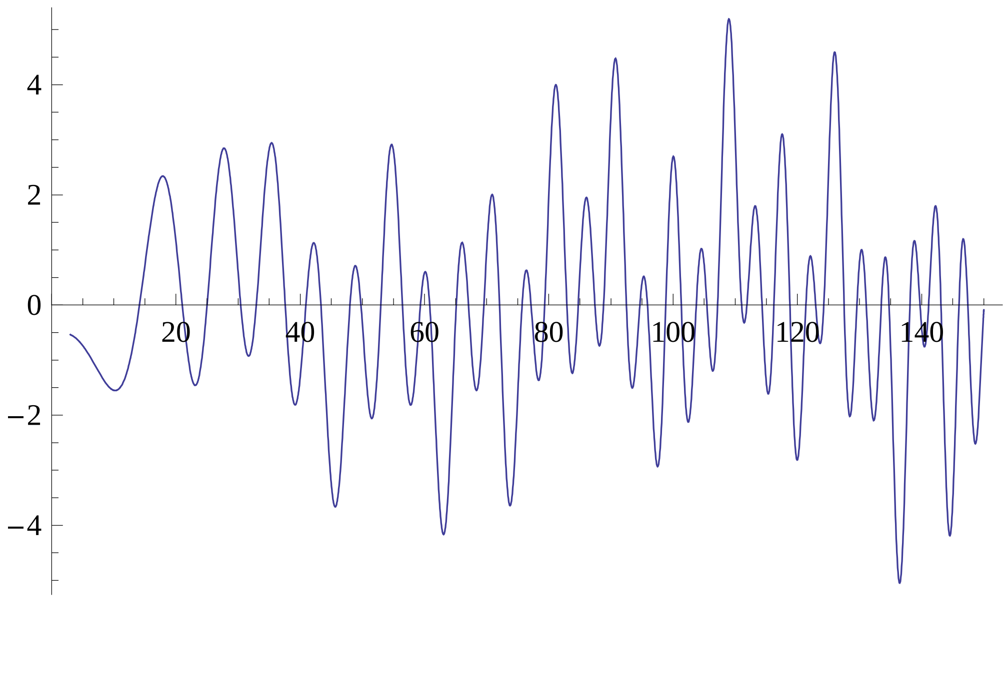
Graph of $\pi(x)-$ Li(x)

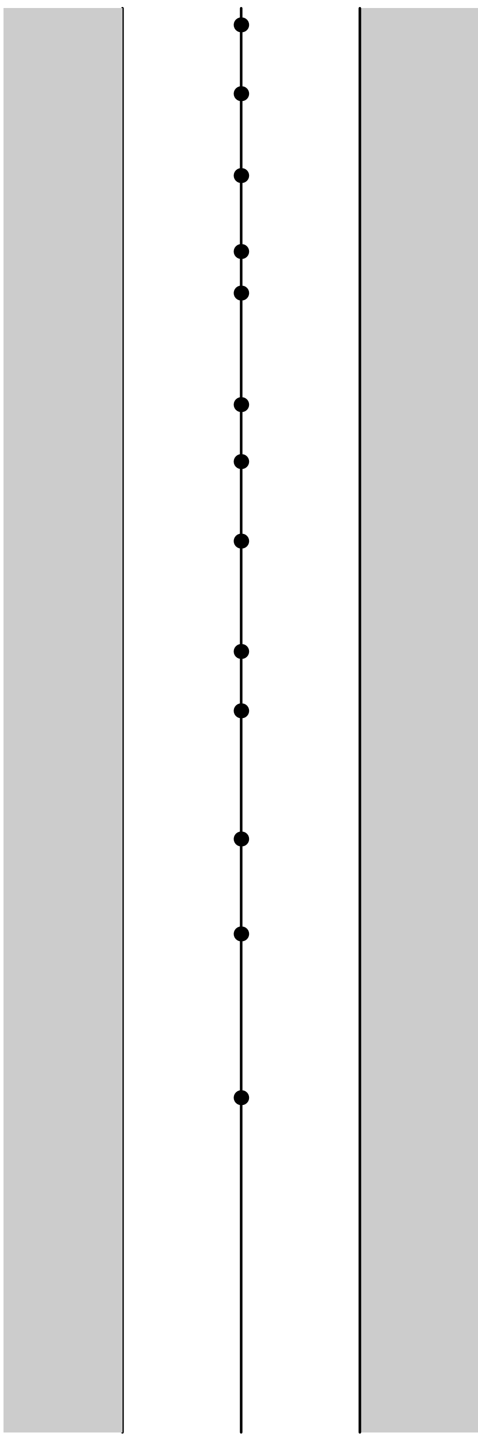
Zeta Function

$$\zeta(s) = \sum_{1}^{\infty} n^{-s} = \prod_{\mathcal{P}} (1 - p^{-s})^{-1}$$

$$\zeta_{\mathbb{Q}}(s) = \pi^{-s/2}\, \Gamma(s/2)\, \zeta(s)$$

$$s \to 1 - s$$





Explicit Formula (Riemann)

$$\pi'(x) = Li(x) - \sum_{\rho} Li(x^{\rho})$$

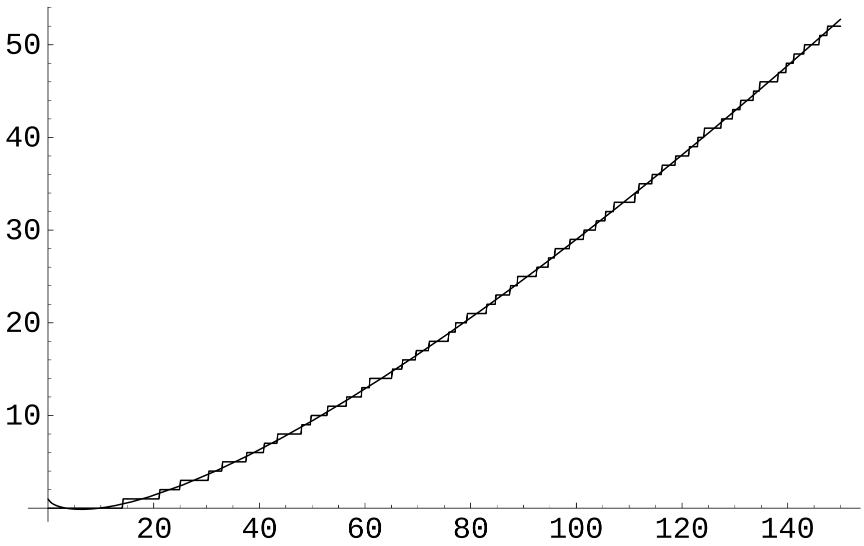
$$+ \int_{x}^{\infty} \frac{du}{u(u^2-1)\log u} + \log \xi(0)$$

$$\pi'(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots$$

Explicit Formula (Weil)

$$\widehat{h}(0) + \widehat{h}(1) - \sum_{\rho} \widehat{h}(\rho) = \sum_{v} \int_{K_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u$$

Quantum Chaos → Riemann Flow ?



$$N(E) = \langle N(E) \rangle + N_{\mathrm{osc}}(E)$$

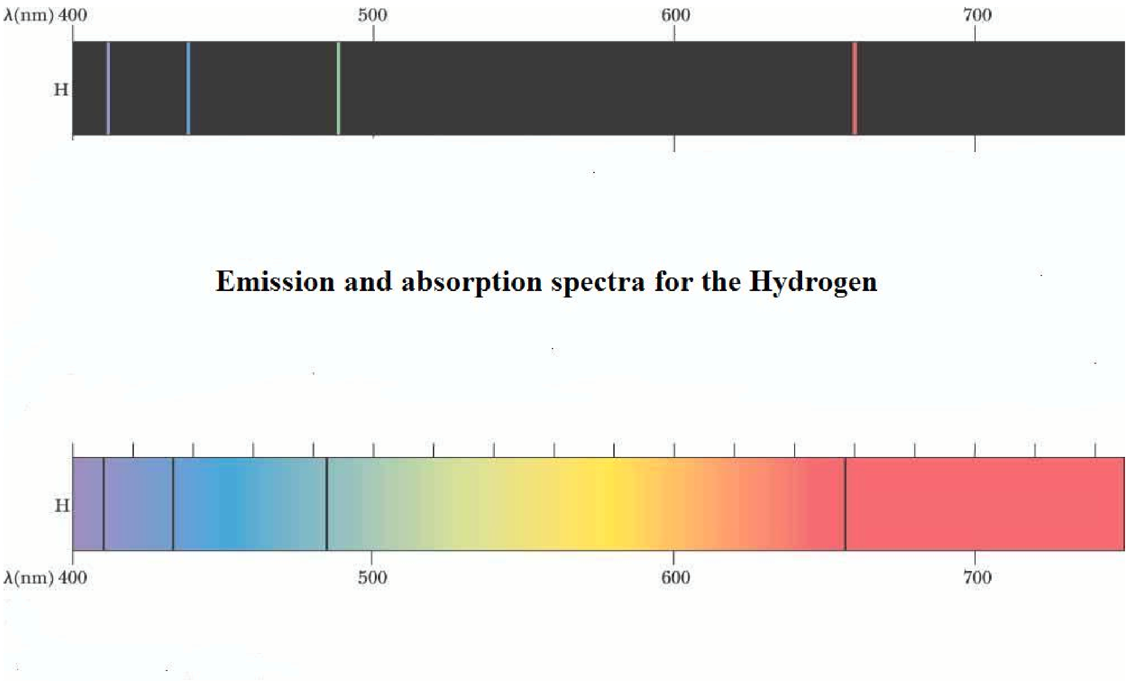
$$\langle N(E) \rangle = \frac{E}{2\pi} \left(\log \frac{E}{2\pi} - 1 \right) + \frac{7}{8} + o(1)$$

Sign Problem :

$$N_{\mathsf{osc}}(E) \sim \frac{-1}{\pi} \sum_{p} \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{p^{m/2}} \sin\left(m\,E\ \log\ p\right)$$

$$N_{\mathsf{osc}}(E) \sim \frac{1}{\pi} \sum_{\gamma_p} \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{2\mathrm{sh}\left(\frac{m\lambda_p}{2}\right)} \sin(m\,E\,T_{\gamma}^{\#})$$

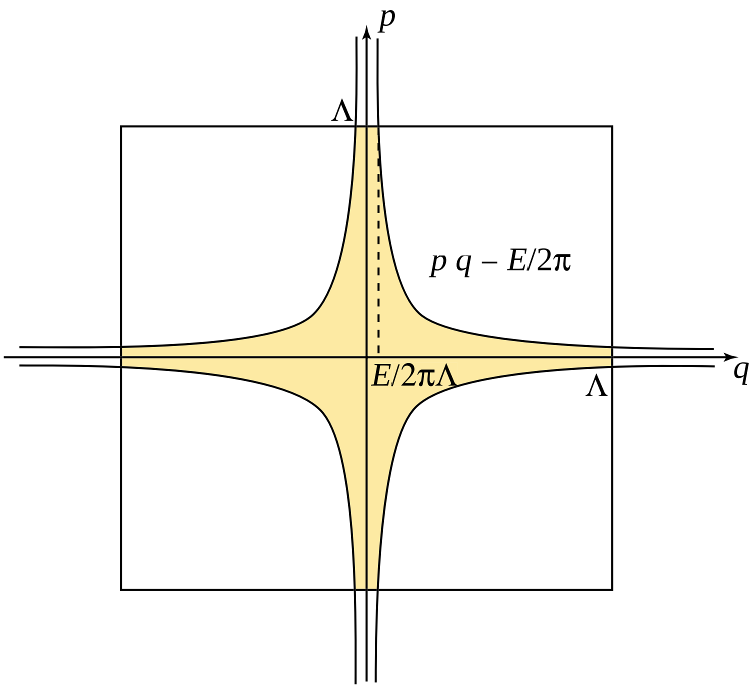
Absorption Spectrum



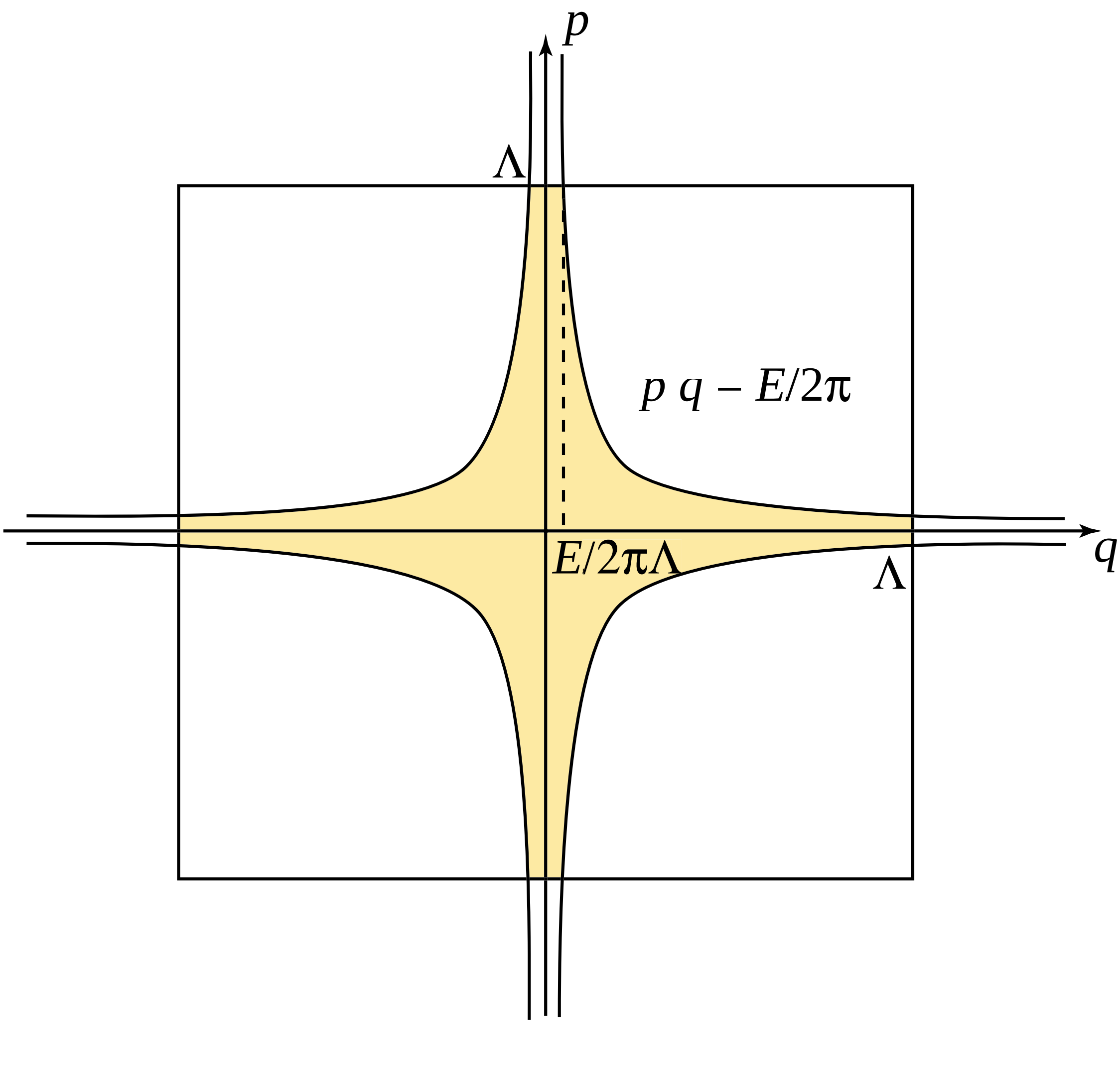
The two kinds of Spectra

$\langle N(E)\rangle$ as symplectic volume $|h| \leq E$

$$h(q,p) = 2\pi q\, p$$



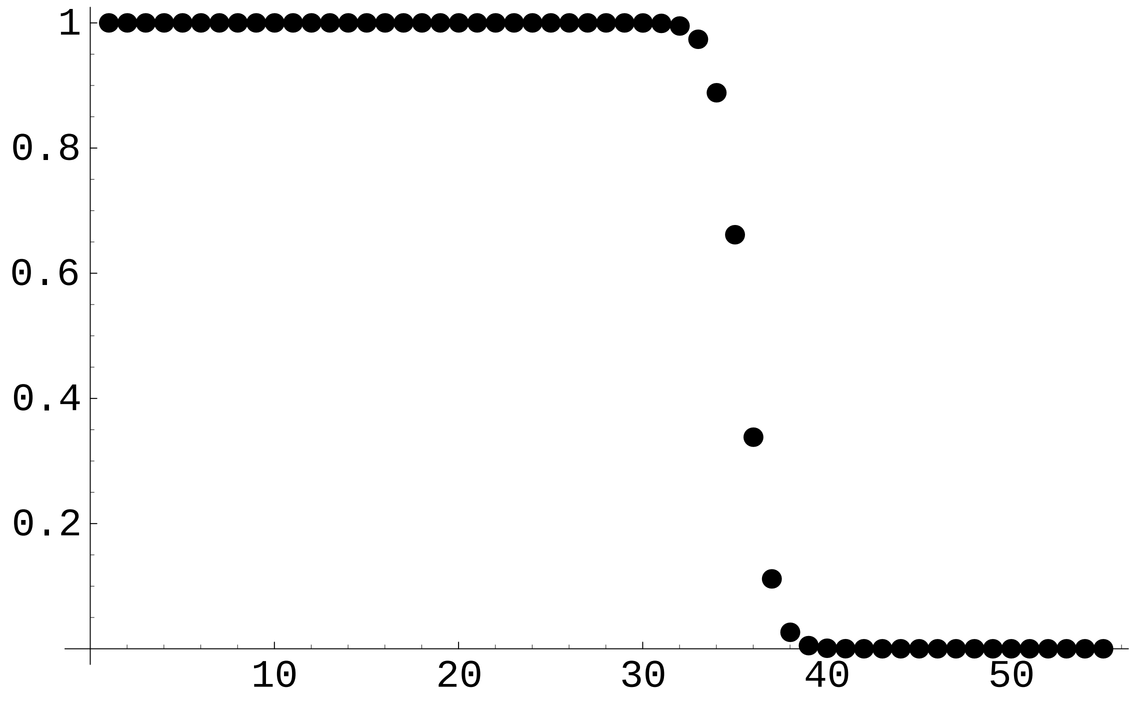
$$\mathrm{Vol}(B_+) = \frac{E}{2\pi} \times 2\log\Lambda - \frac{E}{2\pi}\left(\log\frac{E}{2\pi} - 1\right)$$


p
Λ
p q – E/2π
E/2πΛ
Λ
q

$$P_\Lambda = \{f \in L^2(\mathbb{R})^{even} \mid f(q) = 0, \forall q \text{ with } |q| > \Lambda\}.$$

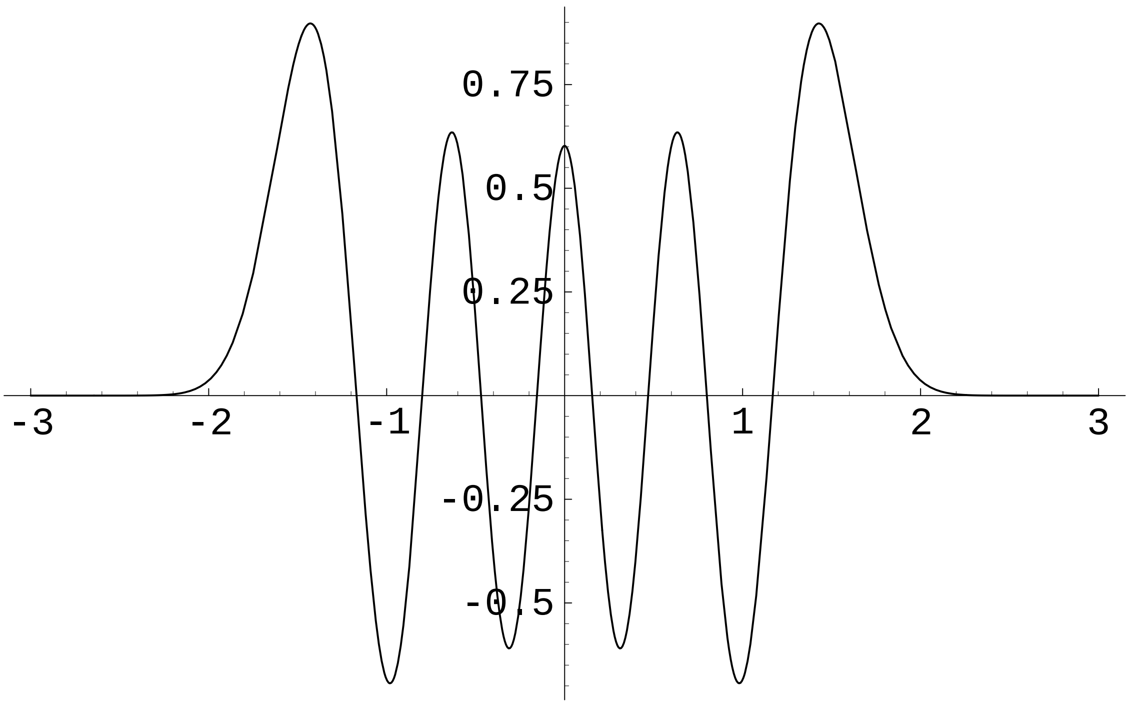
$$\widehat{P}_\Lambda = F P_\Lambda F^{-1}.$$

Pair of projections $\rightarrow$ angle operator



Spheroidal wave functions

Q_Λ near intersection of P_Λ and $\widehat{P}_\Lambda$



Counting the number of quantum states of the Hamiltonian H subject to the constraint $|H| \leq E$ amounts to computing the dimension of the near intersection of the projections Q_Λ and N_E. This is given by $\text{Tr}(Q_\Lambda N_E)$

Theorem

The dimension of the near intersection of Q_Λ with N_E is given by $(\Lambda \to \infty)$

$$\text{Tr}(Q_\Lambda N_E) = \frac{4E}{2\pi} \log \Lambda - 2(\langle N(E) \rangle - 1) + o(1)$$

Let S be a finite set of places with $\infty \in S$. We consider the locally compact ring

$$\mathbb{A}_{\mathbb{Q},S} = \prod_{v \in S} \mathbb{Q}_v.$$

It contains $\mathbb{Q}$ as a subring using the diagonal embedding. We let $\mathbb{Q}_S$ denote the subring of $\mathbb{Q}$ given by rational numbers whose denominator only involves primes $p \in S$. In other words,

$$\mathbb{Q}_S = \{q \in \mathbb{Q} \mid |q|_v \leq 1\,,\ \forall v \notin S\}\,.$$

The group $\mathbb{Q}_S^*$ of invertible elements of the ring $\mathbb{Q}_S$ is of the form

$$\mathbb{Q}_S^* = \mathrm{GL}_1(\mathbb{Q}_S) = \{\pm p_1^{n_1} \cdots p_k^{n_k} \,:\, p_j \in S \smallsetminus \{\infty\}\}.$$

The semi-local adeles class space $X_{\mathbb{Q},S}$ is the quotient

$$X_{\mathbb{Q},S} := \mathbb{A}_{\mathbb{Q},S}/\mathbb{Q}_S^*.$$

$$\mathbb{A}_{\mathbb{Q},S}^* = \mathrm{GL}_1(\mathbb{A}_{\mathbb{Q},S}) = \prod_{p \in S} \mathrm{GL}_1(\mathbb{Q}_p)$$

$$C_{\mathbb{Q},S} = \mathrm{GL}_1(\mathbb{A}_{\mathbb{Q},S})/\mathbb{Q}_S^*$$

Theorem

Let $\mathbb{A}_{\mathbb{Q},S}$ be the semi-local adele class space, with basic character $\alpha = \prod_{v\in S}\alpha_v$. Let $h \in \mathcal{S}(C_{\mathbb{Q},S})$ be a function with compact support. Then, in the limit $\Lambda \to \infty$, one has

$$\mathrm{Tr}(\vartheta(h)R_\Lambda) = 2h(1)\log\Lambda + \sum_{v\in S}\int'_{\mathbb{Q}_v^*}\frac{h(u^{-1})}{|1-u|}\,d^*u + o(1).$$

Spectral realization

Idele class group $\hat{\mathbb{Z}}^* \times \mathbb{R}_+^*$ acts on $L^2(X_{\mathbb{Q}})$ and **zeros of L-functions give the absorption spectrum** with non-critical zeros appearing as resonances.

$$\mathrm{Trace}\,(R_\Lambda\, U(h)) = 2h(1)\log'\Lambda +$$

$$\sum_{v \in S} \int'_{K_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1)$$

$\int'$ is the pairing with the distribution on k_v which agrees with $\frac{du}{|1-u|}$ for $u \neq 1$ and whose Fourier transform relative to α_v vanishes at 1.

Global Trace Formula $\Leftrightarrow$ RH

“L’interdit qui frappe le rêve mathématique, et à travers lui, tout ce qui ne se présente pas sous les aspects habituels du produit fini, prêt à la consommation. Le peu que j’ai appris sur les autres sciences naturelles suffit à me faire mesurer qu’un interdit d’une semblable rigueur les aurait condamnées à la stérilité, ou à une progression de tortue, un peu comme au Moyen Age où il n’était pas question d’écornifler la lettre des Saintes Ecritures. Mais je sais bien aussi que la source profonde de la découverte, tout comme la démarche de la découverte dans tous ses aspects essentiels, est la même en mathématique qu’en tout autre région ou chose de l’Univers que notre corps et notre esprit peuvent connaitre. Bannir le rêve, c’est bannir la source - la condamner à une existence occulte”

$\mathbb{Q}$-**Lattices** (ac + mm)

A $\mathbb{Q}$-lattice in $\mathbb{R}^n$ is a pair (Λ, ϕ), with Λ a lattice in $\mathbb{R}^n$, and

$$\phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda$$

a homomorphism of abelian groups.

Two $\mathbb{Q}$-lattices (Λ_1, ϕ_1) and (Λ_2, ϕ_2) are commensurable if the lattices are commensurable (*i.e.* $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$) and the maps agree modulo the sum of the lattices,

$$\phi_1 \equiv \phi_2 \mod \Lambda_1 + \Lambda_2.$$

$X_{\mathbb{Q}}$ = space of 1-dimensional $\mathbb{Q}$-lattices modulo commensurability.

Consider the quantum statistical mechanical system of two dimensional $\mathbb{Q}$-lattices up to scaling.

1. Each invertible $\mathbb{Q}$-lattice $L = (\Lambda, \phi)$ determines a positive energy representation π_L.
2. The partition function of the system is

$$Z(\beta) = \zeta(\beta)\zeta(\beta - 1)$$

3. For $\beta > 2$, invertible $\mathbb{Q}$-lattices $L = (\Lambda, \phi)$ determine corresponding extremal KMS_β states of the form

$$\varphi_{\beta,L}(f) =$$

$$Z(\beta)^{-1} \sum_{m \in \Gamma \backslash M_2^+(\mathbb{Z})} f(1, m\rho, m(z)) \det(m)^{-\beta}.$$

4. For all $\beta > 2$ the map $L \mapsto \varphi_{\beta,L}$ gives an identification

$$\mathcal{E}_\beta = \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) / \mathbb{C}^*$$

between the set of extremal KMS_β states and the set of invertible $\mathbb{Q}$-lattices up to scaling.

Global field of positive characteristic

k is the field of $\mathbb{F}_q$ valued functions on C.

$$\zeta_k(s) = \prod_{\Sigma_k} (1 - q^{-f(v)s})^{-1}$$

$f(v)$ is the degree of the place $v \in \Sigma_k$.

Functional Equation

$$q^{(g-1)(1-s)}\,\zeta_k(1-s) = q^{(g-1)s}\,\zeta_k(s)$$

where g is the genus of C.

Cohomology and Frobenius

$$\zeta_k(s) = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})}$$

where P is the caracteristic polynomial of the action of the **Frobenius** Fr^* in $H^1_{\mathrm{et}}(\bar{C}, \mathbb{Q}_\ell)$.

The analogue of the Riemann conjecture for global fields of characteristic p means that the eigenvalues of the action of Fr^* in H^1 *i.e.* the complex numbers λ_j of the factorization

$$P(T) = \prod (1 - \lambda_j T)$$

are of modulus $|\lambda_j| = q^{1/2}$.

Proved by Weil (1942) (case $g = 1$ by Hasse)

Frobenius in characteristic zero

(ac + cc +mm)

- **Thermodynamics of noncommutative spaces**
- **Category of Λ-modules = abelian category** (Λ = cyclic category)
- **Endomotives**

Endomotives

A is an inductive limit of reduced finite dimensional commutative algebras over the field $\mathbb{K}$ and S is a semigroup of algebra endomorphisms

$$\rho : A \to A$$

$$\mathcal{A}_{\mathbb{K}} = A \rtimes S$$

Prototype Example :

Endomorphisms of an algebraic variety (group),

$$X_s = \{y \in Y : \; s(y) = *\}.$$

$$X_{sr} \ni y \mapsto r(y) \in X_s.$$

$$X = \varprojlim_{s} X_s$$

$$\xi_{su}(\rho_s(x)) = \xi_u(x)$$

Cooling :

$\mathcal{E}_\beta$ extremal KMS$_\beta$ states, for $\beta > 1$

$$\rho : \mathcal{A} \rtimes_\sigma \mathbb{R} \to \mathcal{S}(\mathcal{E}_\beta \times \mathbb{R}_+^*) \otimes \mathcal{L}^1$$

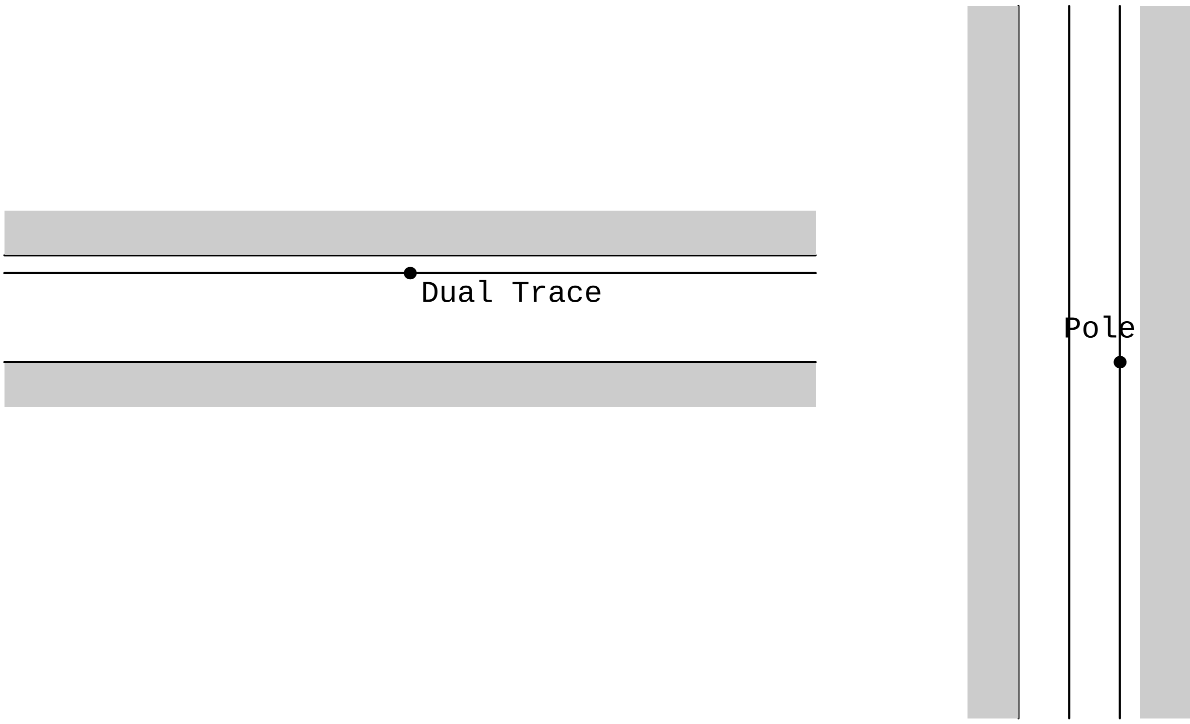
Distillation :

Λ-module $D(\mathcal{A}, \varphi)$ given by the Cokernel of the cyclic morphism given by the composition of ρ with the trace Tr : $\mathcal{L}^1 \to \mathbb{C}$

Dual action :

Spectrum of the canonical action of $\mathbb{R}_+^*$ on the cyclic homology

$$HC_0(\, D(\mathcal{A}, \varphi))$$


Dual Trace
Pole

Explicit Formula = Trace Formula (ac + rm + cc +mm)

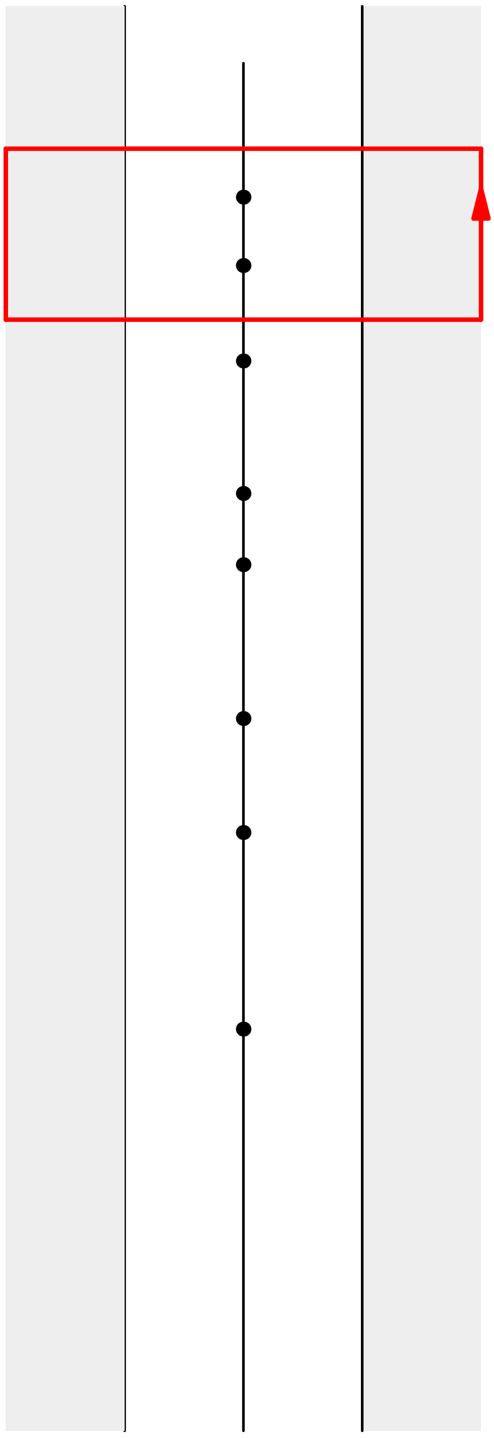
$$\text{Trace}_{H^1}(h) = \widehat{h}(0) + \widehat{h}(1) - \sum_v \int_{K_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u$$

were the last term $\sum_v \int_{K_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u$ is the intersection number

$$Z(h) \bullet \Delta$$

$$\text{Trace}_{H^1}(h) = \widehat{h}(0) + \widehat{h}(1) - \Delta \bullet \Delta\; h(1)$$

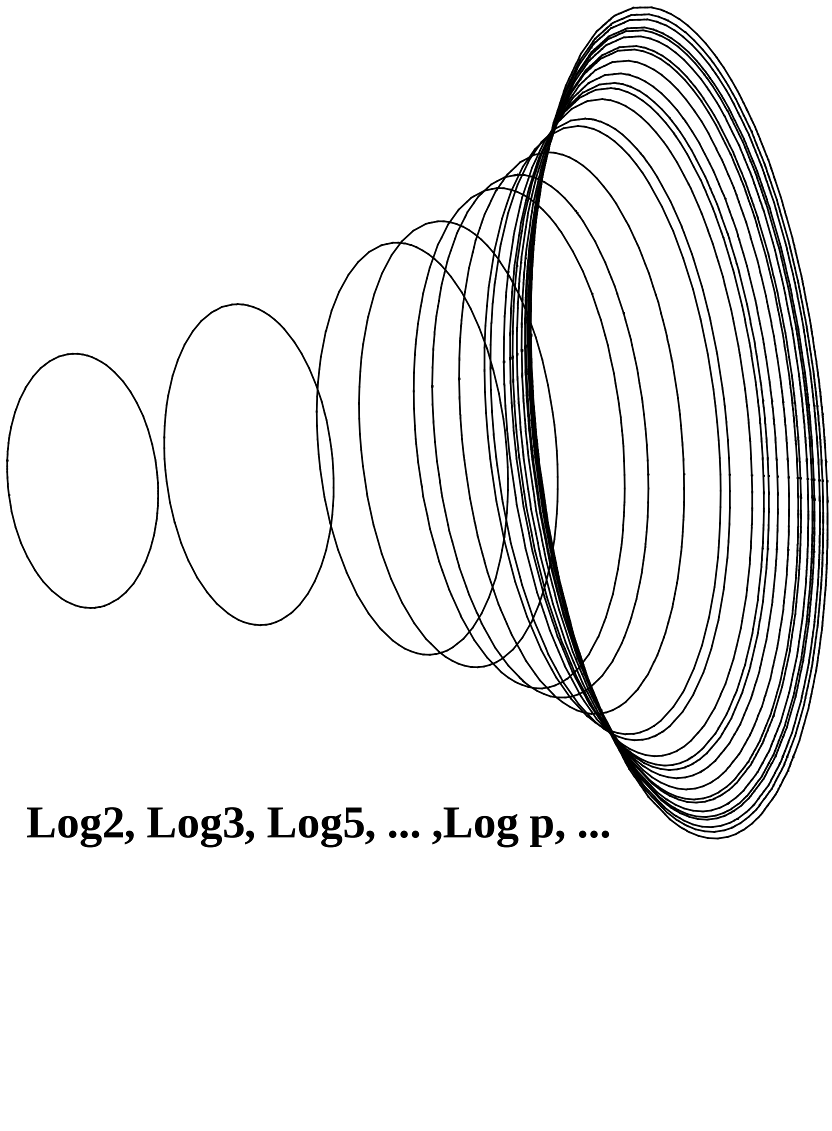
$$- \sum_v \int_{(K_v^*, e_{K_v})} \frac{h(u^{-1})}{|1-u|}\, d^*u$$



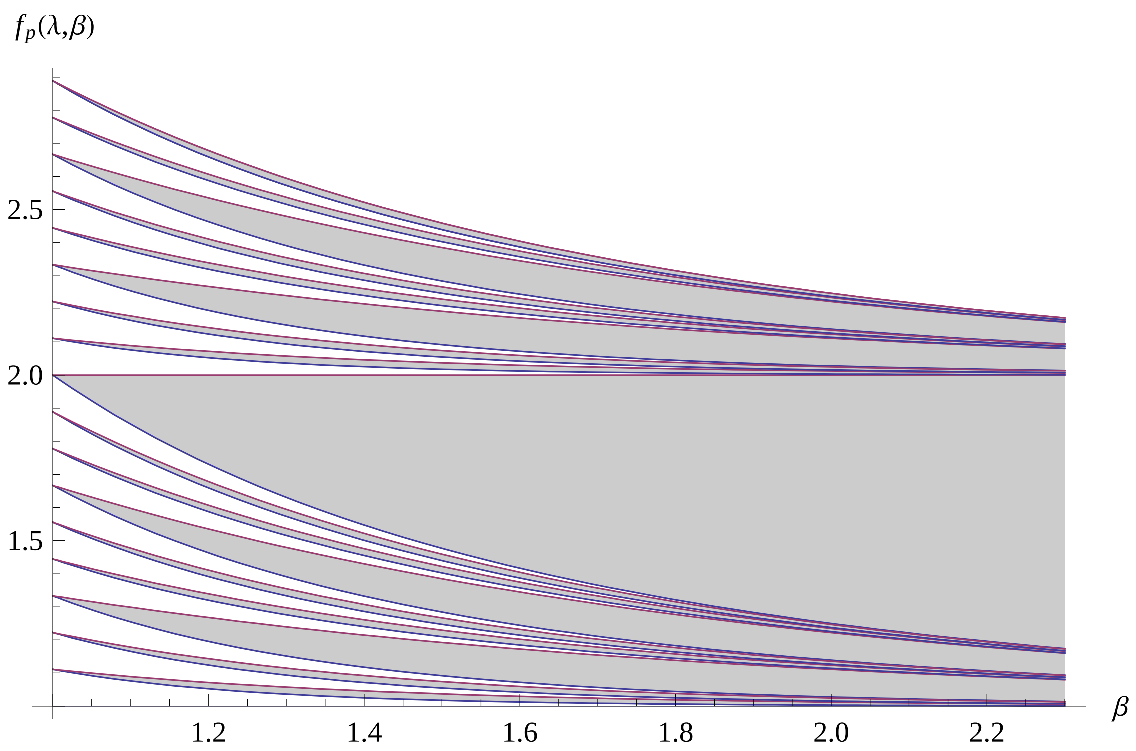
Unramified extensions $K \to K \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$

Analogue for $\mathbb{Q}$ of $K \to K \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$

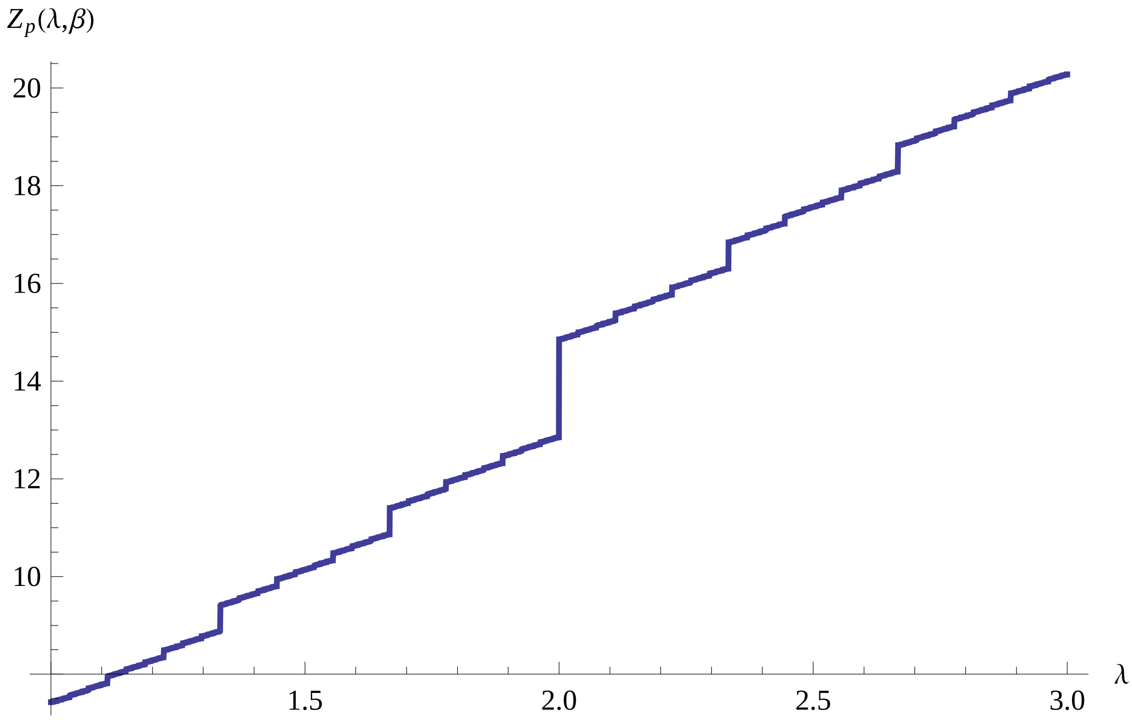
Global field K	**Factor** M
$\mathrm{Mod}\, K \subset \mathbb{R}_+^*$	$\mathrm{Mod}\, M \subset \mathbb{R}_+^*$
$K \to K \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$	$M \to M \rtimes_{\sigma_T} \mathbb{Z}$
$K \to K \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$	$M \to M \rtimes_{\sigma} \mathbb{R}$
Points $C(\bar{\mathbb{F}}_q)$	$\Gamma \subset X_{\mathbb{Q}}$

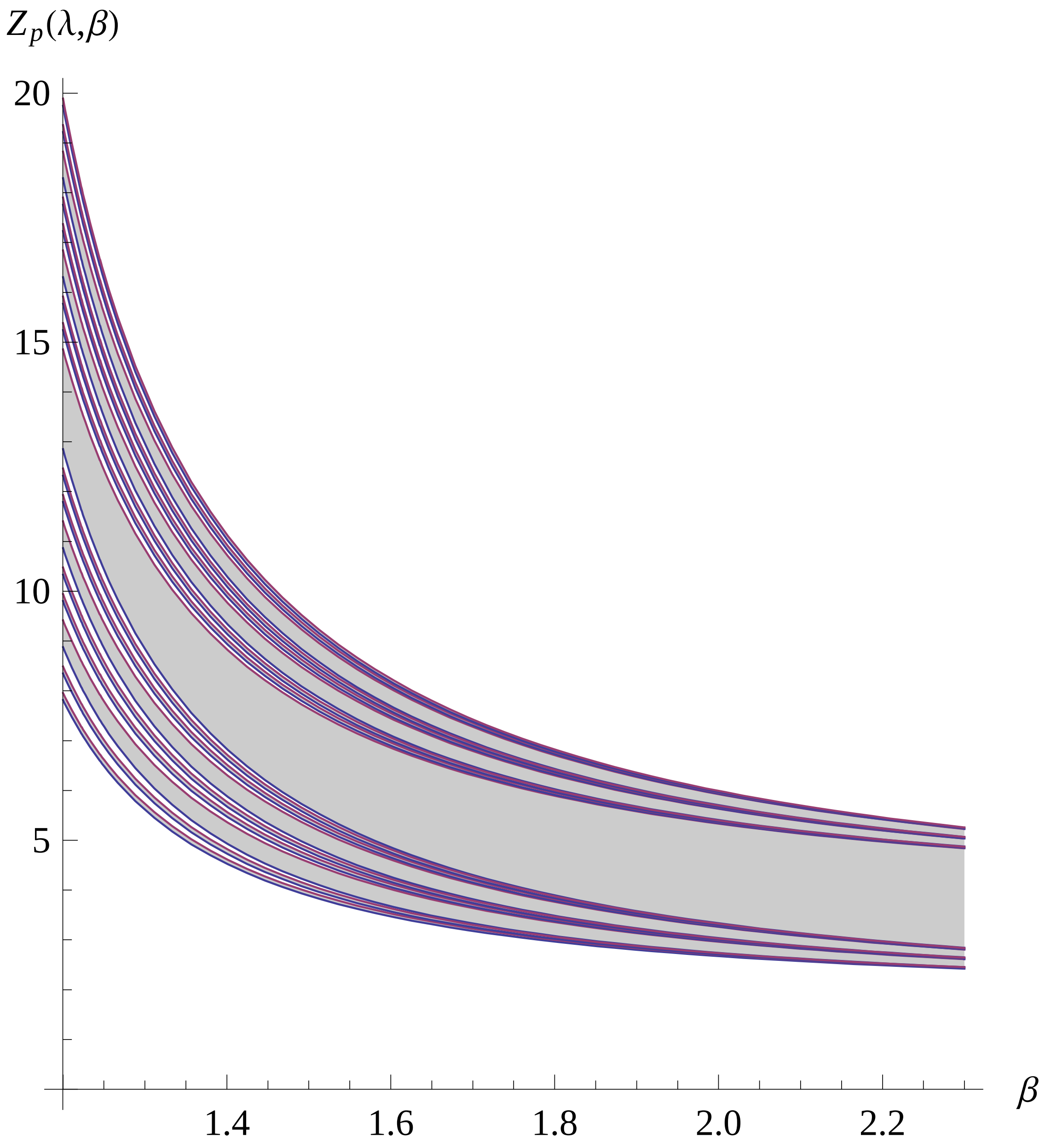


Log2, Log3, Log5, ... ,Log p, ...



$f_p(\lambda,\beta)$
2.5
2.0
1.5
1.2
1.4
1.6
1.8
2.0
2.2
β


$Z_p(\lambda,\beta)$
20
18
16
14
12
10
1.5
2.0
2.5
3.0
λ


$Z_p(\lambda,\beta)$
20
15
10
5
1.4
1.6
1.8
2.0
2.2
β

Weil's proof

The proof of RH rests on two results

– (A) Positivity : Trace$(Z \star Z') > 0$ unless Z is a trivial class.
– (B) Explicit Formula

$$\#\{C(\mathbb{F}_{q^j})\} = \sum (-1)^k \,\mathrm{Tr}(\mathrm{Fr}^{*j} | H^k_{\mathrm{et}}(\bar{C}, \mathbb{Q}_\ell))$$

The role of the positivity condition (A) in Weil's proof is contained in the following :

The following two conditions are equivalent :

- **All L functions with Grössencharakter on K satisfy the Riemann Hypothesis.**
- $\text{Trace}_{H^1}(f \star f^\sharp) \geq 0$ **for all** $f \in \mathcal{S}(C_K)$**.**

$$f \to f^\sharp \,, \qquad f^\sharp(g) = |\,g|^{-1}\,\bar{f}(g^{-1})$$

Weil's proof : Correspondences

$Z\ :\ C \to C,\ P \to Z(P)$

$$U \sim V \Leftrightarrow U - V = (f)$$

$$Z = Z_1 \star Z_2\,, \quad Z_1 \star Z_2(P) = Z_1(Z_2(P))$$

$$Z' = \sigma(Z)$$

$$d(Z) = Z \bullet (P \times C)\,, \quad d^{'}(Z) = Z \bullet (C \times P)$$

Weil defines the *Trace* of a correspondence as follows

$$\text{Trace}(Z) = d(Z) + d^{'}(Z) - Z \bullet \Delta$$

where Δ is the identity correspondence and $\bullet$ is the intersection number.

Proof of positivity (A)

In any (correspondence class)/(trivial ones) one finds a representative Z such that

$$Z > 0\,, \quad d(Z) = g$$

Writing $Z(P) = Q_1 + \cdots + Q_g$, $Z \star Z'(P)$ is the locus of $\sum Q_i \times Q_j$,

$$Z \star Z' = d^{'}(Z)\,\Delta + Y$$

$$Y \bullet \Delta \leq (4\,g - 4)\,d^{'}(Z)\,,$$

$$K(P) = \det\{f_i(Q_j)\}^2$$

$$\Delta \bullet \Delta = 2 - 2\,g$$

$$\text{Trace}(Z \star Z') = 2\,g\,d^{'}(Z) + (2\,g - 2)\,d^{'}(Z) - Y \bullet \Delta$$

$$\geq (4\,g - 2)\,d^{'}(Z) - (4\,g - 4)\,d^{'}(Z) = 2\,d^{'}(Z) \geq 0$$

because $d^{'}(Z) \geq 0$ since Z is effective.

Virtual correspondences	bivariant class Γ
Degree of correspondence	Pointwise index $d(\Gamma)$
deg $D(P) \geq g \Rightarrow \sim$ effective	$d(\Gamma) > 0 \Rightarrow \exists K, \Gamma + K$ onto
Adjusting the degree by trivial correspondences	Fubini step on the test functions
Frobenius correspondence	bivariant element $\Gamma(h)$
Lefschetz formula	bivariant Chern of $\Gamma(h)$ (localization on graph $Z(h)$)