On the classification of Galois objects of $\mathcal{O}_q(SL(2))$

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Reference

In this talk, I will present the results of the following article

- "On the classification of Galois objects over the quantum group of a nondegenerate bilinear form." *Manuscripta Math.*, **122**, (2007), 119–135.

Overlay

1. Definitions
   - Hopf-Galois extensions
   - Bi-Galois objects
   - Fiber functors
   - Cleft Galois objects
   - The Hopf algebra $\mathcal{B}(E)$
   - The algebra $\mathcal{B}(E, F)$
2. Classification results for the Galois objects of $\mathcal{B}(E)$

Non-commutative geometry

Group G	Hopf algebra H

Non-commutative geometry

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Multiplication	Comultiplication

Non-commutative geometry

Group G	Hopf algebra H
Multiplication	Comultiplication
Finite set X with an action $G \times X \to X$	H-comodule algebra Z with a coaction $Z \to H \otimes Z$.

Quantum principal bundle

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$\begin{array}{c} X \\ \downarrow \\ X/G \end{array}$ is a principal fibration	$\mathrm{can} : Z \otimes_B Z \to H \otimes Z$
The fibres are isomorphic to G	is bijective

Canonical map

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Definition

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Definition

- The *covariants elements* $Z^{co\ H}$ of Z are the elements $z \in Z$ such that $\delta(z) = 1 \otimes z$.
- The *canonical map* $\mathrm{can} : Z \otimes_{Z^{co\ H}} Z \to H \otimes Z$ is defined by

$$\mathrm{can}(y \otimes z) = \delta(y)(1 \otimes z)$$

for any $y, z \in Z$.

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- Z is an H-comodule algebra,
- $Z^{co\ H} = B$,
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Generalization of Galois extensions of fields

Example

Let $k \subset L$ be a Galois extension of fields with finite Galois group G. Let k^G be the algebra of functions on G.

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The bijectivity of the canonical map is a consequence of the independance of the characters or, equivalently, of the normal basis property.

Galois objects

Let k be a commutative ring and H a Hopf algebra.

Definition

A faithfully flat (as k-module) Hopf-Galois H-extension of k is called a *Galois object* of H.

Bi-Galois objects

Let k be a commutative ring and H, K be two Hopf algebras.

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Theorem (Schauenburg)

The categories of comodules over H and K are equivalent if and only if there exist a (faithfully flat) H-K-Bi-Galois object.

Functors associated to a Galois object

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Let Z be a Galois object of H. Let us consider the functor

$$\omega_Z : \mathrm{Comod}(H) \longrightarrow \mathrm{Mod}(k)$$

defined by

$$\omega_Z(U) = U \square_H Z,$$

where $\square_H$ stands for the cotensor product over H, that is the equalizer of the coactions of U and Z.

Theorem (Ulbrich)

- *Let Z be a Galois object of H. Then ω_Z is a fiber functor (monoidal exact commuting with colimits).*

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- *Let Z be a Galois object of H. Then ω_Z is a fiber functor (monoidal exact commuting with colimits).*
- *Conversely, if $F : \mathrm{Comod}(H) \to \mathrm{Mod}(k)$ is a fiber functor, it has the form $-\square_H Z$ for a Galois object Z of H.*

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and if $\sigma(h, 1) = \sigma(1, h) = \varepsilon(h)$ for any $h, k, m \in H$.

Twisted comodule algebra

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Let A be a H-comodule algebra and $\sigma : H \otimes H \to k$ be a left cocycle. *The left twisted H-comodule algebra* ${}_\sigma A$ is the H-comodule A together with the product

$$a \cdot_\sigma b = \sigma(a_{(1)}, b_{(1)})\, a_{(2)} b_{(2)}$$

for any $a, b \in A$.

Cleft Galois objects

Let k be a field.

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Definition

Let $\sigma : H \otimes H \to k$ be a left cocycle. Then the H-comodule algebra ${}_\sigma H$ is a Galois object of H and is said to be a *cleft Galois object*.

Finite dimensional Galois objects

Let k be a field.

Theorem

Every Galois object of a finite dimensional Hopf algebra is cleft.

Definition of the algebra $\mathcal{B}(E)$

Let k be a commutative ring, n integer and $E = (E_{ij})_{1 \leq i,j \leq n} \in GL_n(k)$.

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Definition (Dubois-Violette, Launer)

We define $\mathcal{B}(E)$ as the k-algebra generated by n^2 variables a_{ij}, $1 \leq i,j \leq n$, submitted to the matrix relations

$$E^{-1}a^tEa = I_n = aE^{-1}a^tE,$$

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where E^{-1} is the inverse of the matrix E, a the matrix (a_{ij}), I_n the identity matrix of rank n and a^t the transpose of a.

Hopf algebra structure

The algebra $\mathcal{B}(E)$ has a Hopf algebra structure with the comultiplication Δ defined by

$$\Delta(a_{ij}) = \sum_{k=1}^{n} a_{ik} \otimes a_{kj},$$

the counity ε defined by $\varepsilon(a_{ij}) = \delta_{ij}$, for $i, j = 1, \ldots, n$, and where δ_{ij} is the Kronecker symbol, and the antipode S defined by $S(a) = E^{-1} a^t E$.

The Hopf algebra $\mathcal{O}_q(SL(2))$

Let $q \in k^*$ and define E_q the matrix

$$E_q = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix}.$$

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Let $q \in k^*$ and define E_q the matrix

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Then the Hopf algebra $\mathcal{B}(E_q)$ is isomorphic to $\mathcal{O}_q(SL(2))$.

The quantum group of a bilinear form

The Hopf algebras $\mathcal{B}(E)$ where introduced by Dubois-Violette and Launer as "quantum group of a bilinear form". It is the universal Hopf algebra H such that the bilinear form defined by the matrix E is an H-comodule.

Definition of the algebra $\mathcal{B}(E, F)$

Let n, m be intergers and $E \in GL_n(k)$, $F \in GL_m(k)$ be invertible matrices.

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Definition (Bichon)

Let consider the algebra $\mathcal{B}(E, F)$ generated by $n \times m$ variables $z_{ij}, i = 1, \dots, n; j = 1, \dots, m$, submitted to the matrix relations

$$F^{-1} z^t E z = I_m, \quad z F^{-1} z^t E = I_n,$$

where z is the matrix of the generators z_{ij} and I_n, I_m the identity matrices of rank n, m.

Comodule algebra structure

The algebra morphism $\delta : \mathcal{B}(E,F) \to \mathcal{B}(E) \otimes \mathcal{B}(E,F)$, defined by

$$\delta(z_{ij}) = \sum_{k=1}^{n} a_{ik} \otimes z_{kj},$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$, endows $\mathcal{B}(E,F)$ with a left $\mathcal{B}(E)$-comodule algebra.

Bicomodule algebra structure

In the same way, the algebra morphism $\rho : \mathcal{B}(E, F) \to \mathcal{B}(E, F) \otimes \mathcal{B}(F)$ defined by

$$\rho(z_{ij}) = \sum_{k=1}^{n} z_{ik} \otimes b_{kj},$$

where the b_{ij} are the generators of $\mathcal{B}(F)$, endows $\mathcal{B}(E, F)$ with a right $\mathcal{B}(F)$-comodule algebra structure and $\mathcal{B}(E, F)$ is a $\mathcal{B}(E)$-$\mathcal{B}(F)$-bicomodule algebra with bijective canonical maps.

Corepresentations category of $\mathcal{O}_q(SL(2))$

Let k be an algebraicaly closed field and $n \geq 2$ be an integer.

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Theorem (Bichon)

Let $E \in GL_n(k)$ and $q \in k^*$ such that $q^2 + \mathrm{Tr}(E^t E^{-1})q + 1 = 0$. Then we have an equivalence of monoidal categories

$$\mathrm{Comod}(\mathcal{B}(E)) \cong \mathrm{Comod}(\mathcal{O}_q(SL(2))).$$

between the categories of comodules over $\mathcal{B}(E)$ and $\mathcal{O}_q(SL(2))$ respectively.

1 Definitions

2 Classification results for the Galois objects of $\mathcal{B}(E)$
- Classification of Galois objects up to isomorphism
- Classification up to homotopy

Classification up to isomorphism

Let k be a commutative ring and $n \geq 2$ be an integer.

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Theorem

Let $E \in GL_n(k)$ and Z be a Galois object of $\mathcal{B}(E)$. Then there exist $m \geq 2$ and $F \in GL_m(k)$ such that

$$\mathrm{Tr}(F^{-1}F^t) = \mathrm{Tr}(E^{-1}E^t)$$

and such that Z is isomorphic to $\mathcal{B}(E, F)$ as Galois objects of $\mathcal{B}(E)$.

Classification up to isomorphism

Theorem

Let k be a P.I.D, $n, m_1, m_2 \geq 2$ be integers, $E \in GL_n(k)$, $F_1 \in GL_{m_1}(k)$ and $F_2 \in GL_{m_2}(k)$ be matrices such that the algebras $\mathcal{B}(E, F_1)$ and $\mathcal{B}(E, F_2)$ are k-faithfully flat.

Classification up to isomorphism

Theorem

Let k be a P.I.D, $n, m_1, m_2 \geq 2$ be integers, $E \in GL_n(k)$, $F_1 \in GL_{m_1}(k)$ and $F_2 \in GL_{m_2}(k)$ be matrices such that the algebras $\mathcal{B}(E, F_1)$ and $\mathcal{B}(E, F_2)$ are k-faithfully flat.
Then the Galois objects $\mathcal{B}(E, F_1)$ and $\mathcal{B}(E, F_2)$ are isomorphics if and only if $m_1 = m_2$ and if there exists $P \in GL_{m_1}(k)$ such that

$$F_1 = PF_2P^t.$$

Cleft galois objects

Corollary

Let $E \in GL_n(k)$ and $F \in GL_m(k)$ and suppose that $B(E, F)$ is a cleft Galois object of $\mathcal{B}(E)$. Then $m = n$.

Bi-Galois objects

Assume that k is an algebraically closed field of caracteristic zero.

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Corollary

For any $n \geq 2$ and $E \in GL_n(k)$, the group of $\mathcal{B}(E)$-$\mathcal{B}(E)$-Bi-Galois objects is trivial.

Non commutative geometry

Group G	Hopf algebra H
Multiplication	Comultiplication
finit set X endows with an action $G \times X \to X$	H-comodule algebra Z endows with a coaction $Z \to H \otimes Z$.

Quantum principal fiber bundle

The orbits X/G of X under G is a G-set	$B = Z^{co\ H}$ is a sub-algebra and a sub-H-comodule of Z
$X \downarrow X/G$ is a principal fibration	can : $Z \otimes Z \to H \otimes Z$
The fibers are isomorphics to G	is bijective

Homotopy relation

Let k be a commutative ring, H a Hopf algebra, R a k-ring and $k \subset k[t]$ the algebra of polynomial in one inderterinate over k.

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Let k be a commutative ring, H a Hopf algebra, R a k-ring and $k \subset k[t]$ the algebra of polynomial in one inderterinate over k. For any k-module V, we denote $V[t] = V \otimes k[t]$ and for any $i = 0, 1$, we denote $[i] : V[t] \to V$ the linear map sending vt^n to vi^n.

Homotopy relation

Let k be a commutative ring, H a Hopf algebra, R a k-ring and $k \subset k[t]$ the algebra of polynomial in one inderterinate over k. For any k-module V, we denote $V[t] = V \otimes k[t]$ and for any $i = 0, 1$, we denote $[i] : V[t] \to V$ the linear map sending vt^n to vi^n. These two applications induce

$$[i]_* : \mathrm{Gal}_{R[t]}(H[t], k[t]) \to \mathrm{Gal}_R(H, k)$$

for $i = 0, 1$.

Homotoy relation

Definition (Kassel)

Two Galois H-extensions Z_0 and Z_1 of R relatively to the ring k are *homotopically equivalent* if there exists a Galois $H[t]$-extension Z relatively to the ring $k[t]$ such that

$$[i]_*(Z) = Z_i,$$

for $i = 0, 1$.

Homotoy relation

Definition (Kassel)

Two Galois H-extensions Z_0 and Z_1 of R relatively to the ring k are *homotopically equivalent* if there exists a Galois $H[t]$-extension Z relatively to the ring $k[t]$ such that

$$[i]_*(Z) = Z_i,$$

for $i = 0, 1$.
We called homotopy the equivalence relation generated by this relation. We denote $\mathcal{H}_R(H)$ the set of Galois H-extensions of R up to homotopy.

Classification of Galois objects of $\mathcal{O}_q(SL(2))$ up to homotopy equivalence

Theorem

Let k be an algebraically closed ring, m_0, m_1 be integers and F_0, F_1 be invertible matrices of size m_0, m_1 such that their asymetries $\sigma_i = F_i^{-1} F_i^t$ satisfy

$$\mathrm{Tr}(\sigma_i) = -q - q^{-1},$$

for $i = 0, 1$.

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$$\mathrm{Tr}(\sigma_i) = -q - q^{-1},$$

for $i = 0, 1$.
If $m_0 = m_1$ and if the asymetries σ_0 and σ_1 have the same characteristic polynomial, then the two Galois objects $\mathcal{B}(E, F_0)$ and $\mathcal{B}(E, F_1)$ are homotopically equivalent.

Classification of Galois objects of $\mathcal{O}_q(SL(2))$ up to homotopy equivalence

Proposition

Let $m_0, m_1 \geq 2$ be integers, $F_0 \in GL_{m_0}(k)$, $F_1 \in GL_{m_1}(k)$ and suppose that $\mathcal{B}(E_q, F_0)$ and $\mathcal{B}(E_q, F_1)$ are Galois objects of $\mathcal{B}(E_q)$.

Classification of Galois objects of $\mathcal{O}_q(SL(2))$ up to homotopy equivalence

Proposition

Let $m_0, m_1 \geq 2$ be integers, $F_0 \in GL_{m_0}(k)$, $F_1 \in GL_{m_1}(k)$ and suppose that $\mathcal{B}(E_q, F_0)$ and $\mathcal{B}(E_q, F_1)$ are Galois objects of $\mathcal{B}(E_q)$. If $\mathcal{B}(E_q, F_0)$ and $\mathcal{B}(E_q, F_1)$ are homotopic, then the matrices F_0 and F_1 have the same size $m_0 = m_1$.

Corollary

Corollary

Cleft Galois objects of $\mathcal{O}_q(SL(2))$ are homotopically trivial.

Acceptable matrices

Let k be a commutative ring.

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Definition

A matrix $F \in GL_m(k)$ is *acceptable* if $F^{-1}_{mm} = 0$ and if the right most non zero coefficient of the last line of F^{-1}_{mv} is invertible in k.

Acceptable matrices

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Proposition

Let $F \in GL_m(k)$ be an acceptable matrix such that

$$\mathrm{Tr}(F^{-1}F^t) = -q - q^{-1}.$$

Then $\mathcal{B}(E_q, F)$ is a free k-module.

Problem

(P)

Let $F_0, F_1 \in GL_m(k)$ be acceptable matrices such that

$$\mathrm{Tr}(F_0^{-1} F_0^t) = \mathrm{Tr}(F_1^{-1} F_1^t).$$

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Find a matrix $F(t) \in GL_m(k[t])$ such that

1. $F(0) = F_0, \; F(1) = F_1.$

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Find a matrix $F(t) \in GL_m(k[t])$ such that

1. $F(0) = F_0, \; F(1) = F_1.$
2. $\mathrm{Tr}(F(t)^{-1}F(t)^t) = \mathrm{Tr}(F_0^{-1}F_0^t) = \mathrm{Tr}(F_1^{-1}F_1^t).$

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Find a matrix $F(t) \in GL_m(k[t])$ such that

1. $F(0) = F_0, \ F(1) = F_1.$
2. $\mathrm{Tr}(F(t)^{-1}F(t)^t) = \mathrm{Tr}(F_0^{-1}F_0^t) = \mathrm{Tr}(F_1^{-1}F_1^t).$
3. $F(t)$ *is acceptable.*

Reduction of the problem

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Reduction of the problem

The classification of bilinear form ensures that the problem reduce to one of the three following case.

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B. σ_0 is a Jordan bloc of odd dimension d and of eigenvalue 1 (in that case $\sigma_1 = \mathrm{I}_d$),
C. σ_0 is a bloc diagonal matrix, composed of two Jordan blocs of eigenvalues p, p^{-1} and of the same size d (and then σ_1 is diagonal with d coefficients equal to p and d coefficients equal to p^{-1}).

Case A

A) If σ_0 is a jordan bloc of even dimension and of eigenvalue -1 and if there exists and invertible matrix F such that $F^{-1}F^t = \sigma_0$, then F has the following form.

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$$F = \begin{pmatrix} 0 & \cdots & \cdots & 0 & F_{1n} \\ \vdots & & \iddots & -F_{1n} & * \\ \vdots & \iddots & \iddots & * & * \\ 0 & F_{1n} & * & * & * \\ -F_{1n} & * & * & * & * \end{pmatrix}.$$

Solution of the problem in case A

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Case B

B) If σ_0 is a Jordan bloc of odd dimension and of eigenvalue 1 and if there exists an invertible matrix F such that $F^{-1}F^t = \sigma_0$, then F has the following form.

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Case C

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$$\begin{pmatrix} 0 & I_n \\ J_p & 0 \end{pmatrix},$$

where I_n is the identity matrix of size n, J_p is a Jordan bloc of size n and of eigenvlaue p and 0 the zero matrix, has an asymetry equal to σ_0.

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In that case, we can use the matrix $F(t)$

$$\begin{pmatrix} 0 & I_n \\ J_p(t) & 0 \end{pmatrix},$$

where $J_p(t)$ is the matrix with diagonal coefficient equal to p and with coefficient $F(t)_{i,i+1}$ equal to t.