Abstract. We present short and elementary non-geometric analytic proofs of several standard results concerning extension of continuous mappings defined on compacta in \( \mathbb{R}^n \) with values in the unit sphere \( S_{n-1} \).

1. INTRODUCTION. Let \( B_n = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : ||x|| = (\sum_{j=1}^{n} |x_j|^2)^{1/2} \leq 1 \} \) be the Euclidean unit ball and \( S_{n-1} = \partial B_n = \{ x \in \mathbb{R}^n : ||x|| = 1 \} \) the boundary of \( B_n \), usually called the unit sphere\(^1\) in \( \mathbb{R}^n \). It is a direct consequence of Tietze’s extension theorem for functions with values in \([-1, 1]\) that for every closed set \( K \subseteq B_n \) any continuous map \( f : K \to B_n \) admits an extension to a continuous map \( F : B_n \to B_n \) (see below). Our main intention is to give an elementary analytic proof that the same result holds true for \( S_{n-1} \)-valued continuous maps defined on closed subsets of \( S_{n-1} \). All the proofs known to us of this important result are based on general facts from dimension theory in topology. In the same spirit, we also give necessary and sufficient conditions on pairs of compacta \( (K, L) \) in \( \mathbb{R}^n \) such that every continuous map \( f : K \to S_{n-1} \) has a continuous extension to a map \( F : L \to S_{n-1} \). Let us mention that an equivalent result can be found in the classical monograph by Hurewicz and Wallman [2, Theorem VI.12].\(^2\)

2. NOTATION. As usual, \( X^o \) denotes the interior of a set \( X \) in \( \mathbb{R}^n \) and \( X^c = \mathbb{R}^n \setminus X \) its complement. The zero set of a real-valued function \( f : X \to \mathbb{R} \) is denoted by \( Z(f) \).

Let \( C(X, Y) \) be the set of all continuous, vector-valued functions defined on a set \( X \subseteq \mathbb{R}^n \) with target space \( Y \subseteq \mathbb{R}^m \). An \( m \)-tuple \( f = (f_1, \ldots, f_m) \in C(X, \mathbb{R}^m) \) of functions \( f_j \in C(X, \mathbb{R}) \) is said to be invertible, if the components \( f_j \) have no common zeros on \( X \); that is if \( f \in C(X, \mathbb{R}^m \setminus \{(0, \ldots, 0)\}) \). If \( f \) is a real-valued function defined on a set \( X \), then \( ||f||_X := \sup\{||f(x)|| : x \in X\} \).

3. TIEZTE’S EXTENSION THEOREM FOR THE BALL. Let \( K \subseteq B_n \) be closed and suppose that \( f = (f_1, \ldots, f_n) : K \to B_n \) is a continuous map. Obviously \(-1 \leq f_j \leq 1\). Now we may use Tietze’s theorem to extend \( f_j \) to a continuous map \( f_j^* \) on \( B_n \) with values in \([-1, 1]\). Now let

\[
F_j = \frac{f_j^*}{\max\{1, \sqrt{\sum_{k=1}^{n} |f_k^*|^2}\}}.
\]

Then \( F = (F_1, \ldots, F_n) : B_n \to B_n \) is the extension of \( f \) we were looking for.

Can one replace \( B_n \) in the proof above with \( S_{n-1} \)? In fact, no, since the function \( F \) would still have values in \( B_n \), and not in \( S_{n-1} \). But isn’t it possible to divide \( f_j^* \) by

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\(^1\)We use the standard notation \( S_{n-1} \) here, although the symbol \( S_n \) would be more logical. The standard notation comes of course from the fact that \( S_{n-1} \) is an \((n-1)\)-dimensional manifold. But that’s a theorem, and symbols should not be defined based on results, but should be self-contained.

\(^2\)Unfortunately this book is out of print; a reprint at the Dover Publ. series would be welcome.

\[ \sqrt{\sum_{j=1}^{n} |f_j^*|^2} \] instead of taking the max? Well, this could only be done if the functions \( f_j^* \) have no common zeros.

In this connection we have the following standard observation. For shortness, let
\[ |f| := \sqrt{\sum_{j=1}^{n} |f_j|^2} \]
whenever \( f = (f_1, \ldots, f_n) \).

**Lemma 3.1.** Let \( K, L \) be two compact subsets in \( \mathbb{R}^n \) with \( K \subseteq L \). Then a continuous map \( f : K \to \mathbb{R}^n \setminus \{(0, \ldots, 0)\} \) admits a continuous extension \( F : L \to \mathbb{R}^n \setminus \{(0, \ldots, 0)\} \) if and only if \( f/|f| : K \to S_{n-1} \) admits a continuous extension \( F^* : L \to S_{n-1} \).

**Proof.** First we note that the hypothesis on \( f \) implies that \( |f| \) is bounded away from zero. If \( F \) extends \( f \), then \( F/|f| \) extends \( f/|f| \). To prove the converse, let \( F^* : L \to S_{n-1} \) be an extension of \( f/|f| \). Choose any Tietze extension \( T \) of \( |f| \) with \( T(x) \in [\min K |f|, \max K |f|] \). Then \( F \) given by \( F(x) = |f(x)| F^*(x) \) whenever \( x \in K \) and \( F(x) = T(x) F^*(x) \) for \( x \in L \setminus K \) is an extension of \( f \) with \( (0, \ldots, 0) \notin F(L) \).

Thus, studying \( S_{n-1} \)-valued maps is the same as looking at vector-valued functions whose components have no common zeros.

**4. SELF-MAPPINGS OF THE SPHERE.** We begin with the following well-known result from measure theory. For the reader’s convenience, we present a proof, too.

**Lemma 4.1.** Let \( M \subseteq \mathbb{R}^n \) be a compact set of Lebesgue measure zero. Then for any polynomial \( n \)-tuple \( p = (p_1, \ldots, p_n) \) where \( p_j \in \mathbb{R}[x_1, \ldots, x_n] \), \( p(M) \) has measure zero, too.

**Proof.** Let \( \varepsilon > 0 \) and denote the Lebesgue measure of a measurable set \( X \) by \( |X| \). Choose a finite covering of \( M \) with (closed) balls \( B_i \) such that \( \sum_{i=1}^{s} |B_i| < \varepsilon \). Note that \( p(M) \subseteq \bigcup_{i=1}^{s} p(B_i) \).

Let \( x, y \in p(B_i) \). Since \( p \) satisfies a Lipschitz condition on a big ball containing \( M \), we have
\[ ||p(x) - p(y)|| \leq C||x - y|| \]
for some constant \( C > 0 \). Now, for any ball \( B \subseteq \mathbb{R}^n \) and compact set \( K \subseteq \mathbb{R}^n \), there exist constants \( c_1 \) and \( c_2 \) depending only on \( n \) such that
\[ (\text{diam } B)^n = c_1 |B| \quad \text{and} \quad |K| \leq c_2 (\text{diam } K)^n. \]

Hence,
\[ \sum_{i=1}^{s} |p(B_i)| \leq \sum_{i=1}^{s} c_2 (\text{diam } p(B_i))^n \leq c_2 C^n \sum_{i=1}^{s} (\text{diam } B_i)^n \]
\[ = c_2 C^n \sum_{i=1}^{s} c_1 |B_i| \leq c_1 c_2 C^n \varepsilon. \]

We conclude that \( p(M) \) has measure zero. \( \square \)
Lemma 4.2. Let $K \subseteq \mathbb{R}^{n-1}$ be compact and suppose that $f \in C(K, \mathbb{R}^n)$. Then $f$ can be uniformly approximated on $K$ by continuous invertible $n$-tuples.

It is important here that the number of components of the vector-valued function is strictly bigger than the number of variables.

Proof. Let $f = (f_1, \ldots, f_n)$. For $\varepsilon > 0$, choose, according to Weierstrass’ approximation theorem, polynomials $p_j$ with $||p_j - f_j||_K < \varepsilon/2\sqrt{n}$ and let $p = (p_1, \ldots, p_n)$. Now we imbed $K$ into $\mathbb{R}^n$ and consider the compact set

$$K' = \{(x_1, \ldots, x_{n-1}, 0) \in \mathbb{R}^n : (x_1, \ldots, x_{n-1}) \in K\}.$$ 

Moreover, let $\tilde{p}$ be defined by

$$\tilde{p}(x_1, \ldots, x_n) = p(x_1, \ldots, x_{n-1}).$$

Obviously, $\tilde{p}$ is a polynomial $n$-tuple in $\mathbb{R}[x_1, \ldots, x_n]$. Since the $n$-dimensional Lebesgue measure of $K'$ is zero, Lemma 4.1 implies that $\tilde{p}(K') = p(K)$ has measure zero, too. In particular, $p(K)$ has no interior points and so the point $0 \in \mathbb{R}^n$ belongs to the closure of $\mathbb{R}^n \setminus p(K)$.

Hence, for every null-sequence, $(a_i)$, of points in $\mathbb{R}^n \setminus p(K)$, we have that $F^{(i)} := p - a_i$ is an invertible $n$-tuple in $C(K, \mathbb{R}^n)$. Thus, for $i$ sufficiently big,

$$|F^{(i)} - f| = |p - a_i - f| \leq |a_i| + |p - f| < \varepsilon. \quad \blacksquare$$

It is now an interesting observation that the approximation Lemma 4.2 actually yields the following extension Lemma.

Corollary 4.3. Let $K \subseteq \mathbb{R}^{n-1}$ be compact and $B \subseteq \mathbb{R}^{n-1}$ a closed ball with $K \subseteq B^o$. Suppose that $f \in C(K, \mathbb{R}^n)$ is an invertible $n$-tuple. Then $f$ can be extended to an invertible $n$-tuple $F \in C(\mathbb{R}^{n-1}, \mathbb{R}^n)$ with $F \equiv (1, \ldots, 1)$ on $\mathbb{R}^{n-1} \setminus B$.

Proof. Let $h$ be defined as $h(x) = f(x)$ if $x \in K$ and $h(x) = (1, \ldots, 1)$ if $x \in \partial B$. Then $h$ is an invertible $n$-tuple in $C(K \cup \partial B, \mathbb{R}^n)$. Let $H = (H_1, \ldots, H_n) \in C(B, \mathbb{R}^n)$ be a Tietze extension of $h$.

If $H$ is invertible, that is if $\bigcap_{j=1}^n Z(H_j) = \emptyset$, then we are done if we define the invertible $n$-tuple $F$ by $F := H$ on $B$ and $F := (1, \ldots, 1)$ on $\mathbb{R}^n \setminus B$. So assume that $H$ is not invertible.

Choose, again with Tietze, a function $H_{n+1} \in C(B, [0, 1])$ with $H_{n+1} \equiv 0$ on $K \cup \partial B$ and $H_{n+1} \equiv 1$ on $\bigcap_{j=1}^n Z(H_j)$. Then $(H, H_{n+1})$ is an invertible $(n+1)$-tuple in $C(B, \mathbb{R}^{n+1})$. Thus there exist functions $\beta_j \in C(B, \mathbb{R})$, $(j = 1, \ldots, n+1)$, such that

$$\sum_{j=1}^{n+1} \beta_j H_j = 1$$

(just take $\beta_j = H_j/\left(\sum_{i=1}^{n+1} H_i^2\right)$). By Lemma 4.2, we can approximate, as close as we need, $(\beta_1, \ldots, \beta_{n+1})$ on $B$, $B \subseteq \mathbb{R}^{n-1}$, by an invertible $n$-tuple $(u_1, \ldots, u_n)$ in $C(B, \mathbb{R}^n)$. Hence, we may assume that the modulus of
\[ G := \sum_{j=1}^{n} u_j H_j + \beta_{n+1} H_{n+1} \]

is bigger than 1/2 on \( B \).

But \( \beta_{n+1} = \sum_{j=1}^{n} k_j u_j \) for some functions \( k_j \in C(B, \mathbb{R}) \) (just take \( k_j = \beta_{n+1} u_j / \sum_{i=1}^{n} u_i^2 \)). Therefore,

\[ G = \sum_{j=1}^{n} u_j (H_j + k_j H_{n+1}) \]

is zero-free on \( B \). Hence,

\[ F := (H_1 + k_1 H_{n+1}, \ldots, H_n + k_n H_{n+1}) \]

is an invertible \( n \)-tuple in \( C(B, \mathbb{R}^n) \) which extends \( h \in C(K \cup \partial B, \mathbb{R}^n) \). Setting \( F = (1, \ldots, 1) \) on \( \mathbb{R}^n \setminus B \) yields the desired invertible extension of \( f \in C(K, \mathbb{R}^n) \).

**Corollary 4.4.** Let \( K \subseteq S_{n-1} \) be compact. Then every invertible \( n \)-tuple \( f \in C(K, \mathbb{R}^n) \) can be extended to an invertible \( n \)-tuple \( F \in C(S_{n-1}, \mathbb{R}^n) \).

**Proof.** We may assume that \( K \) is not equal to \( S_{n-1} \) and that the north-pole of \( S_{n-1} \) does not belong to \( K \) (otherwise use a suitable rotation).

By an affine transformation we transform the sphere \( S_{n-1} \) to the Riemann sphere \( S \subseteq \mathbb{R}^n \) with center \((0, \ldots, 0, 1/2)\) and radius 1/2. Hence, we think of the \( n \)-tuple \( f \) as being defined on a compact set \( E \) in \( S \) such that the north-pole \( N = (0, \ldots, 0, 1) \) of \( S \) is not in \( E \). Let \( U \subseteq S \) be an open neighborhood of \( N \) (within the topological space \( S \)). Then we look at the stereographic projection \( P \) of \( S \) to the hyperplane \( \mathbb{R}^{n-1} \) and set \( E' = P(E) \). Then \( E' \) is a compact subset in \( \mathbb{R}^{n-1} \) contained in the interior of some ball \( B \). Note that \( \tilde{f} = f \circ P^{-1} \) is an invertible \( n \)-tuple in \( C(E', \mathbb{R}^n) \). By Corollary 4.3, \( \tilde{f} \) can be extended to an invertible \( n \)-tuple \( \tilde{F} \) on \( \mathbb{R}^{n-1} \) with \( \tilde{F} \equiv (1, \ldots, 1) \) on \( \mathbb{R}^{n-1} \setminus B \).

Now \( \tilde{F} \circ P \) is the desired invertible \( n \)-tuple in \( C(S, \mathbb{R}^n) \) extending \( f \in C(E, \mathbb{R}^n) \).

Using Lemma 3.1, we therefore have shown our first intended result.

**Theorem 4.5.** Let \( K \subseteq S_{n-1} \) be compact. Then every continuous map \( f : K \to S_{n-1} \) has an extension to a continuous map \( F : S_{n-1} \to S_{n-1} \).

5. **MORE GENERAL EXTENSION PROBLEMS.**

**Definition 5.1.**

(a) Let \( K \subseteq \mathbb{R}^n \) be compact. Each bounded connected component of \( \mathbb{R}^n \setminus K \) will be called a hole of \( K \).

(b) Let \( K, L \) be two compact sets in \( \mathbb{R}^n \) with \( K \subseteq L \). The pair \((K, L)\) is said to satisfy the hole condition if every hole of \( K \) contains a hole of \( L \).

In this section we present a new proof of the following generalization of Theorem 4.5 (see [2, Theorem VI.12] for a different, but equivalent characterization).
Theorem 5.2. Let $K$ and $L$ be two compact subsets of $\mathbb{R}^n$ with $K \subseteq L$. Then the following assertions are equivalent:

1. every continuous map $f : K \to S^{n-1}$ can be extended to a continuous map $F : L \to S^{n-1}$;
2. the pair $(K, L)$ satisfies the hole condition.

Our proof is based on Theorem 4.5, Zorn’s Lemma for the direction (2) implies (1) and the Brouwer fixed point theorem for (1) implies (2). We need the following lemmas; the first lemma extends Corollary 4.4.

Lemma 5.3. Let $S$ be a sphere in $\mathbb{R}^n$ and let $K \subseteq \mathbb{R}^n$ be compact. Then any continuous map $f : K \to S^{n-1}$ admits a continuous extension $F : K \cup S \to S^{n-1}$.

Proof. If $K \cap S = \emptyset$, then we extend $f$ by setting $F(x) = (1, 0, \ldots, 0) \in S^{n-1} \subseteq \mathbb{R}^n$ if $x \in S$.

If $K \cap S \neq \emptyset$, then by Theorem 4.5, $f|_{K \cap S}$ admits an extension to an invertible $n$-tuple $f^* \in C(S, S^{n-1})$. Now let $F(x) = f(x)$ if $x \in K$ and $F(x) = f^*(x)$ if $x \in S$. Then $F$ is a well defined map on $K \cup S$ and has its values in $S^{n-1}$. Moreover, $F$ extends $f$ and $F$ is continuous.

The proof of the following well known Lemma is a straightforward application of Tietze’s theorem and is left as an exercise to the reader.

Lemma 5.4. Let $M$ be a compact subset of a normal space $X$ and let $f \in C(M, \mathbb{R}^n)$ be an invertible $n$-tuple. Then there is an open set $U$ in $X$ containing $M$ such that $f$ admits an extension to an invertible $n$-tuple in $C(U, \mathbb{R}^n)$.

Lemma 5.5. Let $K \subseteq L \subseteq \mathbb{R}^n$, $K$ and $L$ compact, and suppose that the invertible $n$-tuple $f \in C(K, \mathbb{R}^n)$ cannot be extended to an invertible $n$-tuple in $C(L, \mathbb{R}^n)$. Then there exists a closed set $H$ with $K \subseteq H \subseteq L$ such that

1. $f$ cannot be extended to an invertible $n$-tuple in $C(H, \mathbb{R}^n)$,
2. if $M \supseteq K$ is any proper closed subset of $H$, then $f$ can be extended to an invertible $n$-tuple in $C(M, \mathbb{R}^n)$.

Moreover, any such set $H$ satisfies

(i) $\partial H \subseteq K$,
(ii) $H \setminus K$ is an open set in $\mathbb{R}^n$, and
(iii) $H$ is the union of $K$ with a single hole of $K$.

We will name each such set $H$ an extension hull associated with the pair $(K, L)$. An interesting feature of the Lemma above is that $H$ is independent of $L$ (provided of course that $L$ is a set of non-extendability for some invertible $n$-tuple $f \in C(K, \mathbb{R}^n)$).

Proof. We first show the existence of an extension hull with Zorn’s Lemma. Consider the family $\mathcal{F} = \{H_\lambda : \lambda \in \Lambda\}$ of closed subsets of $L$ satisfying

(a) $K \subseteq H_\lambda \subseteq L$, and
(b) $f$ does not admit an extension to an invertible $n$-tuple in $C(H_\lambda, \mathbb{R}^n)$. 
Note that \( L \in \mathcal{F} \); so \( \mathcal{F} \) is non-void. We endow \( \mathcal{F} \) with the partial ordering \( H_\lambda \prec H_\mu \) if and only if \( H_\lambda \supseteq H_\mu \). Let \( \{ H_y : y \in \Gamma \} \) be an ascending chain; that is for \( \lambda, \mu \in \Gamma \) we either have \( H_\lambda \prec H_\mu \) or \( H_\mu \prec H_\lambda \). We must prove the existence of an upper bound to this chain. In fact, let \( M := \bigcap_{y \in \Gamma} H_y \). Then \( M \) is closed and satisfies (a); that is \( K \subseteq M \subseteq L \). To verify property (b), let us assume, contrariwise, that \( f \) does not admit an extension to an invertible \( n \)-tuple in \( C(M, \mathbb{R}^n) \). By Lemma 5.4, there is an open set \( U \) with \( M \subseteq U \) such that \( f \) admits an extension to an invertible \( n \)-tuple in \( C(U, \mathbb{R}^n) \). The finite intersection property for compacta implies that some \( H_\lambda \) is entirely contained in \( U \). Thus \( f \) would admit an invertible extension to \( H_\lambda \); a contradiction to property (b). Hence, \( M \) is the desired upper bound to our chain. Zorn’s Lemma now implies the existence of a maximal element \( H \in \mathcal{F} \). Hence, \( H \) satisfies (1) and (2) and so \( H \) is the desired extension hull associated with \( (K, L) \).

Next we show that (i) holds, that is \( \partial H \subseteq K \). Suppose, to the contrary, that \( \partial H \not\subseteq K \). Let \( s \in \partial H \setminus K \). Then for any neighborhood \( V \) of \( s \), \( V \setminus H \neq \emptyset \).

Choose an open ball \( B \) in \( \mathbb{R}^n \) centered at a point \( q \in B \setminus H \) close to \( s \) such that \( s \in B \) and \( B \cap K = \emptyset \). Note that \( K \subseteq H \setminus B \subseteq H \). Since \( H \setminus B \) is a proper closed subset of \( H \), property (2) of \( H \) being an extension hull implies that \( f \) admits an invertible extension \( f_1 \in C(H \setminus B, \mathbb{R}^n) \). Let \( S \) be the sphere \( \partial B \). By Lemma 5.3 and Lemma 3.1, \( f_1 \) admits an extension to an invertible \( n \)-tuple \( f_2 \in C((H \setminus B) \cup S, \mathbb{R}^n) \).

Let
\[
p(x) = q + r \frac{x - q}{||x - q||}
\]
be the canonical projection of \( B \setminus \{ q \} \) onto \( \partial B \) where \( r \) is the radius of the ball \( B \). Let
\[
F(x) = \begin{cases} f_2(p(x)) & \text{if } x \in H \cap B \\ f_2(x) & \text{if } x \in H \setminus B. \end{cases}
\]

Note that \( s \in H \cap B \neq \emptyset \). Since \( (H \setminus B) \cap H \cap B \subseteq \partial B = S \), \( F \) is well defined (note that \( q \notin H \cap B \) and that \( p(y) = y \) for \( y \in S \)). Therefore, \( F \) is continuous. Moreover, \( (H \cap B) \cup (H \setminus B) = H \). Thus \( F \) is an invertible \( n \)-tuple in \( C(H, \mathbb{R}^n) \) extending \( f \). This contradicts property (1) of the fact that \( H \) is an extension hull. We deduce that \( \partial H \subseteq K \) and so (i) holds.

It is now a general fact from topology, that \( H \setminus K \) is an open set in \( \mathbb{R}^n \). Indeed, since \( \partial H \subseteq K \),
\[
H \setminus K = H \cap K^c = (H^c \cup \partial H) \cap K^c = H^c \cap K^c.
\]
Thus (ii) holds. Assertion (iii) will be shown later in Corollary 5.7.

A combination of the following Theorem 5.6 with Lemma 3.1 now immediately yields the assertions of Theorem 5.2.

**Theorem 5.6.** Let \( K \) and \( L \) be two compact subsets of \( \mathbb{R}^n \) with \( K \subseteq L \). Then the following assertions are equivalent:

1. every invertible \( n \)-tuple \( f \in C(K, \mathbb{R}^n) \) can be extended to an invertible \( n \)-tuple in \( C(L, \mathbb{R}^n) \), and
2. the pair \( (K, L) \) satisfies the hole condition.

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Proof. We will show the results by contraposition.
For (¬2) implies (¬1), suppose that there is a hole $G$ of $K$ containing no hole of $L$. Since (by definition) $G$ is bounded and connected, we see that $G \subseteq L$. Moreover, $G \subseteq \mathbb{R}^n \setminus K$ and $\partial G \subseteq \partial K$. Fix a point $a \in G$. Let $f$ be defined by

$$f(x) = x - a \quad \text{for} \quad x \in K.$$  

Then $f$ is an invertible $n$-tuple in $C(K, \mathbb{R}^n)$ but it cannot be extended to an invertible $n$-tuple in $C(L, \mathbb{R}^n)$. In fact, suppose to the contrary that $F_1 \in C(L, \mathbb{R}^n)$ is an invertible $n$-tuple extending $f$. Let

$$F_2(x) = \begin{cases} x - F_1(x) & \text{if } x \in \overline{G} \\ a & \text{if } x \in \mathbb{R}^n \setminus G. \end{cases}$$

Since

$$(\mathbb{R}^n \setminus G) \cap \overline{G} \subseteq \partial G \subseteq \partial K \subseteq K,$$

we see that $F_2$ is well-defined, because on this intersection both expressions are equal. Since $F_2$ is bounded, there is a closed ball $B \subseteq \mathbb{R}^n$ with $\overline{G} \subseteq B$ such that $F_2$ is a continuous self-map of $B$. By Brouwer’s fixed point theorem (see for example [10]) there is a point $w \in B$ with $F_2(w) = w$. Because $a \in G$, the second case in the definition for $F_2(w)$ is not possible. Hence, $w \in \overline{G} \subseteq L$ and $w = F_2(w) = w - F_1(w)$. Thus $F_1(w)$ would be the zero vector, a contradiction to the invertibility of the $n$-tuple $F_1$ on $L$. We conclude that $f$ cannot be extended to an invertible continuous $n$-tuple on $L$. Hence, (−1) holds.

For (−1) implies (−2), by hypothesis, we suppose that there is an invertible $n$-tuple $f \in C(K, \mathbb{R}^n)$ which cannot be extended to an invertible $n$-tuple in $C(L, \mathbb{R}^n)$. According to Lemma 5.5, let $H \subseteq L$ be an extension hull associated with the pair $(K, L)$. Then $H \setminus K$ is an open set. We claim that $\partial (H \setminus K) \subseteq K$. To see this, we just note that $H \setminus K$ is open and so, due to the fact that $H$ is closed,

$$\partial (H \setminus K) \subseteq H \setminus (H \setminus K) = H \cap K \subseteq K.$$  

Now let $G$ be a component of $H \setminus K$. In particular, $G \subseteq H$ and $\partial G \subseteq \partial (H \setminus K)$. Since $H$ is compact, $G$ is bounded. We show that $G$ actually is a hole of $K$. In fact, as a connected subset of $\mathbb{R}^n \setminus K$, $G$ is contained in a component $C$ of $\mathbb{R}^n \setminus K$ (that may be the unbounded one). If $G \neq C$, then there is a path $\gamma$ in $C$ joining a given point $u \in G$ to some point $v \in C \setminus G$. But then there is a boundary point $w \in \partial G$ belonging to $\gamma$. Hence,

$$w \in C \cap \partial G \subseteq C \cap \partial (H \setminus K) \subseteq C \cap K = \emptyset.$$  

This contradiction shows that $G = C$.

On the other hand, we have that

$$G \subseteq H \setminus K \subseteq L \setminus K \subseteq L.$$  

Thus we have found a hole of $K$ entirely contained in $L$. Thus (−2) holds.

As a corollary we are now able to prove assertion (iii) in Lemma 5.5.

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Corollary 5.7. Let $H$ be an extension hull associated with the pair $(K, L)$, where $L$ is a set of non-extendability for some invertible $n$-tuple $f \in C(K, \mathbb{R}^n)$. Then $H$ is the union of $K$ with a single hole of $K$.

Proof. By the proof above, each component of the open set $H \setminus K$ is a hole of $K$. But $H \setminus K$ cannot have 2 components; in fact, let $\Omega_1$ and $\Omega_2$ be two components of $H \setminus K$. Then $K \cup \Omega_2 \subseteq H$ is a proper closed subset of $H$. Since $H$ is assumed to be an extension hull, each invertible $n$-tuple in $C(K, \mathbb{R}^n)$ would then be extendable to an invertible $n$-tuple in $C(K \cup \Omega_2, \mathbb{R}^n)$. By Theorem 5.6, the pair $(K, K \cup \Omega_2)$ then satisfies the hole condition, which is impossible, because $\Omega_2$ is a hole of $K$ entirely contained in $K \cup \Omega_2$.

Corollary 5.8. Let $K \subseteq \mathbb{R}^n$ be a compact set for which $\mathbb{R}^n \setminus K$ is connected. Then any invertible $n$-tuple $f = (f_1, \ldots, f_n) \in C(K, \mathbb{R}^n)$ admits an extension to an $n$-tuple $F = (F_1, \ldots, F_n)$ in $C(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$0 < m \leq \sum_{j=1}^{n} |f_j|^2 \leq M < \infty,$$

where $m = \min_K \sum_{j=1}^{n} |f_j|^2$ and $M = \max_K \sum_{j=1}^{n} |f_j|^2$.

Proof. Let $B$ be a closed ball centered at the origin containing $K$. Recall that $|f| = \sqrt{\sum_{j=1}^{n} |f_j|^2}$. Let $T_{|f|}$ be any Tietze extension of $|f|$ from $K$ to $B$ satisfying $\sqrt{m} \leq |T_{|f|}| \leq \sqrt{M}$.

Since $K$ has no holes, we may apply Theorem 5.6 together with Lemma 3.1 to get a map $U : B \to S_{n-1}$ such that $U|_K = f/|f|$. Now let

$$V(x) = T_{|f|}(x)U(x) \text{ if } x \in B.$$

Then $V$ is an invertible $n$-tuple in $C(B, \mathbb{R}^n)$ that extends $f$. Finally, if $r$ is the radius of $B$, let

$$F(x) = \begin{cases} V(x) & \text{if } ||x|| \leq r \\ V\left(\frac{x}{||x||}\right) & \text{if } ||x|| \geq r. \end{cases}$$

Then $F$ is the desired invertible $n$-tuple extending $f$ with $m \leq |F|^2 \leq M$.

If $\mathbb{R}^n \setminus K$ is no longer connected, then we have to increase the dimension of the target space to obtain the following variant of Corollary 4.3.

Corollary 5.9. Let $K \subseteq \mathbb{R}^n$ be compact. Then any invertible $(n+1)$-tuple $f = (f_1, \ldots, f_{n+1}) \in C(K, \mathbb{R}^{n+1})$ admits an extension to an $(n+1)$-tuple $F = (F_1, \ldots, F_{n+1})$ in $C(\mathbb{R}^n, \mathbb{R}^{n+1})$ such that

$$0 < m \leq \sum_{j=1}^{n+1} |F_j|^2 \leq M < \infty,$$

where $m = \min_K \sum_{j=1}^{n+1} |f_j|^2$ and $M = \max_K \sum_{j=1}^{n+1} |f_j|^2$.
Proof. Embed $K$ by the map $\iota: (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$ into $\mathbb{R}^{n+1}$ and apply Corollary 5.8. Note that $\mathbb{R}^{n+1} \setminus \iota(K)$ is automatically connected. 

Summary. In each of the following cases every continuous mapping can be extended to a continuous mapping on the larger set on the left-hand side, the target space remaining the same:

\[
\begin{align*}
K \subseteq B_n & \quad \rightarrow & \quad B_n, \\
K \subseteq S_{n-1} & \quad \rightarrow & \quad S_{n-1}, \\
K \subseteq \mathbb{R}^{n-1} & \quad \rightarrow & \quad S_{n-1}, \\
K \subseteq \mathbb{R}^n & \quad \rightarrow & \quad S_{n-1}, \\
\mathbb{R}^n \setminus K \text{ conn.} & \quad \rightarrow & \quad S_{n-1}.
\end{align*}
\]

Readers interested in generalizations of the subject of this note are referred to the monographs [1, 2, 8, 9] dealing with abstract dimension theory and the role played there by extension-mappings. Applications of this theory are to be found in the theory of stable ranks for Banach algebras (see [12]); for example Lemma 4.2 tells us that the topological stable rank of $C(K, \mathbb{R})$ is less than or equal to $n$ whenever $K \subseteq \mathbb{R}^{n-1}$, $K$ compact.

The methods of this note stem from various research articles developed by the authors (see, e.g., [11], [4, 5, 6, 7] and [3]).

REFERENCES


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Triangles Don’t Tile the Plane

James P. Conlan

This is a do-it-yourself proof that the standard tiling of the plane by equilateral triangles cannot be accomplished by congruent, mutually disjoint sets. Each triangle, in this case, consists of the union of the interior of a closed triangle with some subset of the boundary of the triangle. The “tiling” is a point set tiling having no gaps or overlaps.

Consider triangle A in the figure. The number “1” indicates that the upper vertex of triangle A covers the vertex point. The other triangles adjacent to this vertex must be labeled “0” since they cannot contain the vertex.

Triangle B must have some vertex labeled “1”. By symmetry, without loss of generality, label the upper vertex “1”. The adjacent vertices must be labeled “0”. This forces a “1” in triangle C. Keep going. The labeling of vertices is forced in triangles B through M. A contradiction appears after M. The plane cannot be tiled with identical triangles.