STRONG UNIQUENESS SETS AND \(t\)-ANALYTIC SETS
FOR \(H^\infty\) AND \(H^\infty + C\)

by

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Abstract. — We determine the relations of the \(t\)-analytic sets for the algebra \(H^\infty\) of bounded holomorphic functions in the unit disk with those in the Sarason algebra \(H^\infty + C\) and give a description of the strong uniqueness sets for these algebras.

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1. Introduction

In this note we introduce the notion of strong uniqueness sets for Banach function algebras \(A\) and compare this class of sets with the recently introduced \(t\)-analytic sets for \(A\). Recall that a subset \(E\) of the spectrum \(M(A)\) of \(A\) is said to be \(t\)-analytic, (denoted by \(E \in \mathcal{A}\)), if for every \(f \in A\) and every open set \(U\) in \(M(A)\) with \(U \cap E \neq \emptyset\) one has \(f \equiv 0\) on \(E\) whenever \(f \equiv 0\) on \(E \cap U\). For example the empty set and every singleton is a \(t\)-analytic set. Also, each point in \(M(A)\) is contained in a maximal, though not necessary unique, \(t\)-analytic set (see [2]).

A nonvoid set \(E \subseteq M(A)\) is called a uniqueness set for \(A\), if for every \(f\) and \(g\) in \(A\), \(f = g\) whenever \(f\) and \(g\) coincide on \(E\). If this property also holds locally, that is, if for every open set \(U\) in \(M(A)\) with \(U \cap E \neq \emptyset\), \(f|_{U \cap E} = g|_{U \cap E}\) implies \(f = g\), then we say that \(E\) is a strong uniqueness set for \(A\). The set of strong uniqueness sets for \(A\) is denoted by \(\mathcal{U}\).

It is clear that any strong uniqueness set is a \(t\)-analytic set. These classes are different though, since for example a singleton \(\{x\}\) is a strong uniqueness set if and only if \(M(A) = \{x\}\) (and so \(A = C(\{x\}) \cong \mathbb{C}\)). We remark that a

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$t$-analytic set $E$ is a strong uniqueness set if and only if the hull-kernel closure $\hat{E}$ of $E$ equals $M(A)$. Recall that $\hat{E}$ is the zero set (or hull) of the ideal $I(E, A) = \{ f \in A : \text{f}|E \equiv 0 \}$.

The concept of $t$-analytic sets, originally considered only for open sets in $[1]$ in connection with local/restricted decomposability of multiplication operators on commutative, semisimple Banach algebras, was first given in this generality in $[2]$. It turned out that it has a surprising connection to closed prime ideals: if $E \subseteq M(A)$ is a $t$-analytic set for $A$, then the ideal $I(E, A)$ is a closed prime ideal. We also unveiled the connection of $t$-analytic sets with ideals of the form $J(x, A) = \{ f \in A : \text{f vanishes identically on a neighborhood of } x \text{ in } M(A) \}$, that appear in problems on spectral synthesis for Banach function algebras (see for example $[5]$). In fact, if $E \in \mathcal{A}$ and $x \in E$, then $E$ is contained in the zero set $k_{A}(x)$ of the ideal $J(x, A)$.

A description of the $t$-analytic sets in concrete algebras seems to be a very hard problem. At the moment, such a characterization is only known for the disk-algebra and general regular function algebras (see $[2]$). In the present paper we will be concerned with the $t$-analytic sets in the algebra $H^{\infty}$ of bounded analytic functions in the open unit disk $D$ and its associated Sarason algebra $H^{\infty} + \mathbb{C}$ of sums of boundary values of functions in $H^{\infty}$ and (complex-valued) continuous functions on the unit circle $T$. We assume that the reader is familiar with the structure of the maximal ideal spaces of these algebras (see $[4]$).

First results in this direction were given in $[2]$. For example, it is known $[2]$ that in $H^{\infty} + \mathbb{C}$ the $t$-analytic sets are very small. In fact, if $E$ is a $t$-analytic set for $H^{\infty} + \mathbb{C}$ then, due to the fact that $E \subseteq k_{H^{\infty} + \mathbb{C}}(x)$ for some $x \in E$, the set $E$ is nowhere dense and contained in a single fiber.

The situation for $H^{\infty}$ is quite different. Here $k_{H^{\infty}}(x)$ equals $M(H^{\infty})$ for every $x \in M(H^{\infty})$. Moreover, $t$-analytic sets for $H^{\infty}$ may be big. For example, the unit disk $D$ is a $t$-analytic set for $H^{\infty}$. Hence, by the corona theorem, the whole spectrum $M(H^{\infty})$ is the maximum $t$-analytic set for $H^{\infty}$. But also the Shilov boundary, $\partial H^{\infty}$, of $H^{\infty}$ is a $t$-analytic set for $H^{\infty}$. Or the uniqueness set $[0, 1]$. On the other hand, the uniqueness set $[0, 1] \cup \{-1/2\}$ is not $t$-analytic. Neither the corona $M(H^{\infty}) \setminus D$ is a $t$-analytic set for $H^{\infty}$.

A way for comparing $t$-analytic sets for $H^{\infty}$ with those of $H^{\infty} + \mathbb{C}$ comes from the fact that the spectrum $M(H^{\infty} + \mathbb{C})$ of $H^{\infty} + \mathbb{C}$ can be identified with the corona $M(H^{\infty}) \setminus D$ of $D$ in $M(H^{\infty})$. Also, the Shilov boundaries for $H^{\infty}$ and $H^{\infty} + \mathbb{C}$ coincide and can be identified with $M(L^{\infty})$, the maximal ideal space of the algebra of (equivalence classes) of Lebesgue measurable and essentially bounded functions on $T$. 
Natural questions now arise. For instance, which \( t \)-analytic sets for \( H^\infty + C \) are \( t \)-analytic for \( H^\infty \)? Are there essentially other \( t \)-analytic sets for \( H^\infty \) besides those mentioned above? Can we describe all the strong uniqueness sets for \( H^\infty \), respectively \( H^\infty + C \)?

In this paper we give answers to these questions.

We conclude the introduction with some additional notations used throughout the paper. For a Banach function algebra \( A \), we always consider \( A \) as a set of continuous functions that live on \( M(A) \).

If \( f \in A \), then \( Z(f) = \{ x \in M(A) : f(x) = 0 \} \) is the zero set of \( f \). If \( I \) is an ideal in \( A \), then \( Z(I) = \bigcap_{f \in I} Z(f) \) is the zero set (or hull) of \( I \). The interior of a subset \( E \) of a topological space \( X \) will be denoted by \( E^o \); its closure by \( \overline{E} \). If \( X \subseteq M(A) \), then \( Z_X(f) = Z(f) \cap X \).

2. Some general facts on strong uniqueness sets

It is known that \( E \) is a \( t \)-analytic set for \( A \) if and only if the closure \( \overline{E} \) of \( E \) is one (see [2]). The same property is valid for the class of strong uniqueness sets:

\[ \text{Proposition 2.1.} \quad \text{Let } A \text{ be a Banach function algebra. Then } E \subseteq M(A) \text{ is a strong uniqueness set for } A, \text{ that is } E \in \mathcal{U}, \text{ if and only if } \overline{E} \in \mathcal{U}. \]

\[ \text{Proof.} \quad \text{Assume that } E \in \mathcal{U}. \text{ Let } U \subseteq M(A) \text{ be open, } U \cap \overline{E} \neq \emptyset, \text{ and let } f \equiv 0 \text{ on } U \cap \overline{E}. \text{ Then } U \cap E \neq \emptyset \text{ and } f \equiv 0 \text{ on } U \cap E. \text{ Hence } f \text{ is the zero function. Thus } \overline{E} \in \mathcal{U}. \]

Conversely, let \( \overline{E} \in \mathcal{U} \). Let \( U \subseteq M(A) \) be open, \( U \cap \overline{E} \neq \emptyset \), and \( f \equiv 0 \) on \( U \cap \overline{E} \). Then the openness of \( U \) and the continuity of \( f \) imply that \( f \equiv 0 \) on \( U \cap E \). Hence \( f \) is the zero function, too. Thus \( E \in \mathcal{U} \). \( \square \)

As an immediate consequence we have

\[ \text{Corollary 2.2.} \quad \text{Let } E \subseteq F \subseteq \overline{E}. \text{ Then } F \in \mathcal{U} \text{ whenever } E \in \mathcal{U}. \]

\[ \text{Observation 2.3.} \quad \text{Let } E \in \mathcal{U} \text{ and suppose that } F \text{ is closed. Then } E \setminus F \in \mathcal{U} \cup \{ \emptyset \}. \text{ In other words, every nonvoid relatively open subset of a strong uniqueness set belongs to } \mathcal{U}, \text{ too.} \]

Whereas the set \( \mathcal{O} \) of \( t \)-analytic sets always contains the empty set and the singletons, its subset \( \mathcal{U} \) of strong uniqueness sets may be void. Indeed, this happens quite frequently, as the following result shows.

\[ \text{Theorem 2.4.} \quad \text{If the set of strong uniqueness sets for a Banach function algebra is not empty, then the spectrum of } A \text{ is connected.} \]
Proof. — We show the contraposition. Suppose that $M(A)$ is disconnected. Then there are at least two disjoint open-closed sets $S_1$ and $S_2$ such that $S_1 \cup S_2 = M(A)$. Let $E \subseteq M(A)$. Without loss of generality, we may assume that $E \cap S_2 \neq \emptyset$. By Shilov’s idempotent theorem, (see [3, p. 88]), there is a function $f \in A$ such that $f \equiv 1$ on $S_1$ and $f \equiv 0$ on $S_2$. Now we choose $U = S_2$. Then $U$ is open, $f \equiv 0$ on $U \cap E$, but $f$ is not the zero function. Hence $E \notin \mathcal{U}$.

Of course the connectivity condition above is far from being sufficient for $\mathcal{U}$ to be non-empty. In fact, $\mathcal{U} = \emptyset$ for any regular algebra strictly containing $\mathbb{C}$. But $\mathcal{U}$ may be empty, too, for other algebras with connected spectrum, as for example $H^\infty + \mathbb{C}$ (see Theorem 3.1).

Lemma 2.5. —

1. If $M(A)$ is not a strong uniqueness set, then there exists $x \in M(A)$ such that $k_A(x) \nsubseteq M(A)$.
2. If $\mathcal{U} = \emptyset$, then the set of points $x$ for which $k_A(x) \nsubseteq M(A)$, is dense in $M(A)$.

Proof. — (1) Since $M(A) \notin \mathcal{U}$ there exists a nonvoid open set $V \subseteq M(A)$ and $f \in A$ such that $f \equiv 0$ on $V$ but $f \neq 0$. Hence, for any $x \in V$, we have $k_A(x) \nsubseteq M(A)$.

(2) Let $\emptyset \neq V$ be open in $M(A)$. Since $\mathcal{U} = \emptyset$, $V \notin \mathcal{U}$. Hence there is a second open set $V'$ such that $V \cap V' \neq \emptyset$ and a non-constant function $f \in A$ such that $f \equiv 0$ on $V \cap V'$. Thus any $x \in V \cap V'$ has the property that $k_A(x) \nsubseteq M(A)$.

Proposition 2.6. — If for every $x \in M(H^\infty)$, $k_A(x)$ is a proper subset of $M(A)$, then $\mathcal{U} = \emptyset$.

Proof. — Let $E \subseteq M(A)$. Choose $x \in E$. Since $k_A(x) \neq M(A)$, there exists $y \in M(A) \setminus k_A(x)$ and a function $f \in J(x, A)$ with $f(y) \neq 0$. Hence $f \equiv 0$ on $E \cap Z(f)^c \neq \emptyset$, but $f \neq 0$. Therefore, $E$ is not a strong uniqueness set.

 Whereas the union of two $t$-analytic sets is, in general, not $t$-analytic, (even if they are non-disjoint and connected), (see [2]), we have the following result concerning the subclass of strong uniqueness sets.

Proposition 2.7. — Any union of strong uniqueness sets in a Banach function algebra is a strong uniqueness set again.

Proof. — Let $E_\alpha \in \mathcal{U}$, and set $E = \bigcup E_\alpha$. Note that strong uniqueness sets are never empty. Let $U$ be open and suppose that $f \equiv 0$ on $U \cap E$. We assume that this last set is non empty. Hence there exists $\alpha$ such that $U \cap E_\alpha \neq \emptyset$. Since $f \equiv 0$ on $U \cap E_\alpha$, our hypothesis implies that $f \equiv 0$. Thus $E \in \mathcal{U}$.
The class $\mathcal{U}$, though, is not stable with respect to intersections; even if those intersections are non-empty. For example, $[-1,0]$ and $[0,1]$ are strong uniqueness sets for the disk-algebra $A(\mathbb{D})$, but their intersection not.

**Corollary 2.8.** — Let $A$ be a Banach function algebra for which $\mathcal{U} \neq \emptyset$. Then there exists a maximum strong uniqueness set.

Note that in the class of $t$-analytic sets for $A$ there always exist maximal elements; but, in general, no maximum $t$-analytic set (see [2]).

In [2, Example 2.4], an example of a compact set $K = K_1 \cup K_2 \subseteq \mathbb{C}$ is given which shows that for the algebra $A = A(K)$ of all functions continuous on $K$ and holomorphic in the interior $K^\circ$ of $K$, $K_1$ and $K_2$ are (non-disjoint) maximal $t$-analytic sets. Moreover, $k_A(z) = K = M(A)$ for every $z \in K_1$ and $k_A(z) = K_2 \subseteq M(A)$ for any $z \in K_2 \setminus K_1$. Here we can now add that $K_1$ is the maximum strong uniqueness set for $A(K)$ (see figure 1).

![Figure 1. An instructive example](image)

**Corollary 2.9.** — Let $A$ be a Banach function algebra for which $\mathcal{U} \neq \emptyset$. Then the biggest strong uniqueness set, $E_{\text{max}}$, is also a maximal $t$-analytic set.

**Proof.** — Obviously $E_{\text{max}}$ is $t$-analytic. Now let $E_{\text{max}} \subseteq E$ for some $t$-analytic set $E$. We show that $E \in \mathcal{U}$. Let $U$ be open, $U \cap E \neq \emptyset$, and suppose that $f \equiv 0$ on $U \cap E$. Since $E$ is $t$-analytic, $f \equiv 0$ on $E_{\text{max}}$. In particular $f \equiv 0$ on $E_{\text{max}}$. But $\hat{E_{\text{max}}} = M(A)$. Hence $f \equiv 0$ and so $E \in \mathcal{U}$. The maximality of $E_{\text{max}}$ now implies that $E = E_{\text{max}}$.  

\[ \square \]
3. Strong uniqueness sets and \( t \)-analytic sets for \( H^\infty + C \)

We start with the following results from [2]; except item 4. Recall that a thin point \( x \in M(H^\infty) \setminus \mathbb{D} \) is any point lying in the \( M(H^\infty) \)-closure of a sequence \((z_n) \in \mathbb{D}^\mathbb{N} \) satisfying

\[
\lim_{j \to \infty} \prod_{n \neq j} \rho(z_n, z_j) = 1,
\]

where \( \rho(z, w) = |(z-w)/(1-\overline{z}w)| \) is the pseudohyperbolic distance. Moreover, \( P(x) \) is the Gleason part associated with a point \( x \in M(H^\infty + C) \). The zero sets \( k_{H^\infty + C}(x) \) of the ideals \( I(x, H^\infty + C) \) are called \( k \)-hulls and will be denoted by \( k(x) \). See [5, 7, 8] for a detailed study of these \( k \)-hulls.

**Theorem 3.1.**
1. Let \( E \) be a \( t \)-analytic set for \( H^\infty + C \) and suppose that \( x \in E \). Then \( E \subseteq k(x) \).
2. If the \( t \)-analytic set \( E \) meets the Shilov boundary of \( H^\infty + C \), then \( E \) is a singleton.
3. If \( E \) is a maximal \( t \)-analytic set containing the thin point \( x \), then \( E = P(x) \).
4. There are no strong uniqueness sets for \( H^\infty + C \).

**Proof.** (4) This follows from Proposition 2.6 and the fact that for each \( x \in M(H^\infty + C) \), \( k(x) \neq M(H^\infty + C) \) (see [5]).

It is conjectured that in \( H^\infty + C \) all maximal \( t \)-analytic sets and all hull-kernel closed \( t \)-analytic sets with cardinal bigger than 2 are given by the closures of Gleason parts (see [2]).

4. \( t \)-analytic sets for \( H^\infty \)

In [2] it was implicitly shown that in the disk-algebra the class of \( t \)-analytic sets with cardinal bigger than two and the class of strong uniqueness sets coincide. In \( H^\infty \), the class of \( t \)-analytic sets containing more than one point is much bigger than \( \mathcal{U} \). For instance, the closure of any non-trivial Gleason part in the corona of \( H^\infty \) is \( t \)-analytic, but obviously not a uniqueness set (see [2]).

However, if the set \( E \) belongs to the Shilov-boundary, \( \partial H^\infty \), of \( H^\infty \), then the result just mentioned for \( A(\mathbb{D}) \) remains valid.

**Proposition 4.1.** A nonvoid set \( E \subseteq \partial H^\infty \) is \( t \)-analytic for \( H^\infty \) if and only if \( E \) is either a singleton or a strong uniqueness set.

**Proof.** One direction being obvious, we need only show that every \( t \)-analytic set \( E \) with \( E \subseteq \partial H^\infty \) and containing at least two points is a strong uniqueness
set for $H^\infty$. In fact, by [2], the ideal $I(E, H^\infty)$ is a closed prime ideal. By [9, Theorem 3.3], any non-zero closed prime ideal whose hull meets the Shilov boundary, is maximal. Thus $I(E, H^\infty) = \{0\}$ whenever $E$ contains at least two points. Hence $E$ is a strong uniqueness set in that case. □

In what follows, let $\hat{\sigma}$ denote the lifted Lebesgue measure defined on the Borel sets of the extremely disconnected set $M(L^\infty)$ (see [3, p. 17]). Recall that for any $f \in L^\infty$, 

$$\int_T f \, d\sigma = \int_{M(L^\infty)}  f \, d\hat{\sigma},$$

and that $\hat{\sigma}(B^c) = \hat{\sigma}(B) = \hat{\sigma}(\overline{B})$ for any Borel set $B \subseteq M(L^\infty)$. Here $\hat{f}$ is the Gelfand transform of $f \in L^\infty$. The characteristic function of a set $S \subseteq \mathbb{T}$ is denoted by $\chi_S$. Similarly for sets in $M(L^\infty)$. It is well known that the sets 

$$\{\hat{\chi}_S = 1\} := \{x \in M(L^\infty) : \hat{\chi}_S(x) = 1\},$$

$S \subseteq \mathbb{T}$ Lebesgue-measurable, form a basis of closed-open sets for the topology on $M(L^\infty)$ (see [3, p. 17]).

Let $QC$ be the algebra of quasi-continuous functions on $\mathbb{T}$; that is $QC$ is the biggest $C^*$-subalgebra of $H^\infty + C$. Moreover, let $QA = QC \cap H^\infty$. See [10, 11] for a thorough study of these algebras.

The following Lemma has been communicated to me by Keiji Izuchi.

**Lemma 4.2.** — Let $E$ be a nonvoid closed subset of $M(L^\infty)$ with $\hat{\sigma}(E) = 0$. Then there exists a non-constant function $f \in H^\infty$ such that $f \equiv 0$ on $E$.

**Proof.** — Let $K_n$ be a sequence of closed-open sets in $M(L^\infty)$ satisfying 

$$E \subseteq K_{n+1} \subseteq K_n$$

and $\hat{\sigma}(K_n) \to 0$. Let 

$$F = \sum_{n=1}^\infty (1 - \chi_{K_n})/n^2.$$

Then $F \in C(M(L^\infty))$. Hence there is $q \in L^\infty$ such that $\hat{q} = F$. Moreover, 

$$F \equiv 0 \text{ on } P := \bigcap_{n=1}^\infty K_n.$$

Note that $E \subseteq P$ and that $\hat{\sigma}(P) = 0$. By Wolff [11, Theorem 1], there is a nonzero $f \in QA$ such that $f F \in QC$. Then, with $X = M(L^\infty)$,

$$Z_X(f) = Z_X(f) \cup Z_X(F) = Z_X(f) \cup P.$$

Since $Z_X(f) \cup P$ has lifted Lebesgue measure 0, we deduce from [11, Lemma 2.3] that $Z_X(f) \cup P$ is a weak peak interpolation set for QA. Hence there is a non-constant $g \in QA$ such that $g \equiv 0$ on $Z_X(f) \cup P \supseteq E$. □
Theorem 4.3. — Let E be a nonvoid closed subset of ∂H∞. The following assertions are equivalent:

1. E is a strong uniqueness set for H∞;
2. For every Lebesgue measurable set S ⊆ T with strictly positive Lebesgue measure either \( \hat{\sigma}(E \cap \{\hat{\chi}_S = 1\}) > 0 \) or \( E \cap \{\hat{\chi}_S = 1\} = \emptyset \).

In particular, \( \partial H^\infty \in \mathcal{U} \).

Proof. — (2) ⇒ (1): Let \( U \subseteq M(H^\infty) \) be any open set with \( U \cap E \neq \emptyset \). Let \( x \in U \cap E \). Then there is a Lebesgue measurable set \( S \subseteq T \) with \( \sigma(S) > 0 \) such that

\[
x \in \{\hat{\chi}_S = 1\} \subseteq U \cap M(L^\infty).
\]

Hence \( \emptyset \neq E \cap \{\hat{\chi}_S = 1\} \subseteq E \cap U \). Suppose that for some \( f \in H^\infty \backslash \{0\} \), \( Z(f) \cap \partial H^\infty \) has lifted Lebesgue measure 0. Our hypothesis that \( \hat{\sigma}(E \cap \{\hat{\chi}_S = 1\}) > 0 \) now implies that \( f \) is the zero function in \( H^\infty \). Hence \( E \in \mathcal{U} \).

(1) ⇒ (2) will be proven via contraposition. So suppose \( E \subseteq \partial H^\infty = M(L^\infty) \) satisfies \( E \cap \{\hat{\chi}_S = 1\} \neq \emptyset \), but

\[
\hat{\sigma}(E \cap \{\hat{\chi}_S = 1\}) = 0
\]

for some measurable set \( S \subseteq T \) of positive Lebesgue measure. By Lemma 4.2, there is a non-constant \( f \in H^\infty \) with \( f \equiv 0 \) on \( E \cap \{\hat{\chi}_S = 1\} \). Hence \( E \) cannot be a strong uniqueness set for \( H^\infty \).

Is it possible to give a description of the strong uniqueness sets \( E \) in \( \partial H^\infty \) using only properties of \( H^\infty \) when viewed as a set of functions defined on \( T \)?

For example let \( S \) be a measurable subset of \( T \). Then \( S \) is a ‘strong uniqueness set’ for \( H^\infty|_T \) if and only if \( \sigma(S \cap I) > 0 \) for every open arc \( I \subseteq T \) with \( S \cap E \neq \emptyset \). Which relations can one expect between \( S \) and \( E \)?

Next we compare the \( t \)-analytic sets for \( H^\infty \) and \( H^\infty + C \).

Lemma 4.4. — Let \( x \in M(H^\infty + C) \). Denote the identity function on \( T \) by \( z \). Then the \( k \)-hull \( k(x) \) of \( x \) is contained in a single fiber

\[
M_\lambda = \{m \in M(H^\infty + C) : m(z) = \lambda\},
\]

\(|\lambda| = 1\).

Proof. — The assertion follows from the facts that \( k(x) \) is contained in a unique \( C(T) \)-level set

\[
E_\lambda = \{m \in M(H^\infty + C) : m(f) = f(\lambda) \text{ for every } f \in C(T)\}
\]

which coincides with the fibers.
We shall need several times the following Lemma, whose first assertion is a special case of [5, Lemma 2.4].

**Lemma 4.5.** — Let \( x \in M(H^\infty + C) \setminus \partial H^\infty \). Then the ideal \( J(x, H^\infty + C) \) is algebraically generated by Blaschke products. Moreover, \( k(x) \) is hull-kernel closed in \( H^\infty \).

**Proof.** — Since \( J := J(x, H^\infty + C) \) is generated by Blaschke products, we have that \( k(x) = \bigcap_{B \in J} Z(B) \). Accordingly, for every \( y \in M(H^\infty) \setminus k(x) \), there exists a Blaschke product \( B \in J \) with \( B \equiv 0 \) on \( k(x) \), but \( B(y) \neq 0 \). Thus \( k(x) \) is hull-kernel closed in \( H^\infty \).

**Lemma 4.6.** — Let \( E \subseteq M(H^\infty) \setminus \mathbb{D} \) be a \( t \)-analytic set for \( H^\infty \) and let \( x \in E \setminus \partial H^\infty \). Then \( E \subseteq k(x) \).

**Proof.** — By Lemma 4.4, \( k(x) \) is contained in a single fiber. In particular, \( M(H^\infty + C) \setminus k(x) \neq \emptyset \). So let \( y \in M(H^\infty + C) \setminus k(x) \). Hence \( E \cap U \neq \emptyset \). Choose an open set \( U \) in \( M(H^\infty + C) \) with \( x \in U \). Note that \( E \cap U \neq \emptyset \). Choose an open set \( V \) in \( M(H^\infty) \) such that \( U = V \cap M(H^\infty + C) \). Then \( B \equiv 0 \) on \( E \cap V = E \cap U \). Since \( E \) is \( t \)-analytic for \( H^\infty \), we conclude that \( y \notin E \). Hence \( E \subseteq k(x) \).

**Theorem 4.7.** — Let \( E \subseteq M(H^\infty) \setminus \mathbb{D} \) be a \( t \)-analytic set for \( H^\infty \). Then \( \overline{E} \) either is entirely contained in the Shilov boundary or in \( M(H^\infty + C) \setminus \partial H^\infty \).

**Proof.** — Assume that there is \( x \in E \setminus \partial H^\infty \). By Proposition 2.1, \( \overline{E} \) is \( t \)-analytic. Hence, by Lemma 4.6, \( \overline{E} \subseteq k(x) \). By [5], \( k(x) \cap \partial H^\infty = \emptyset \). Thus \( \overline{E} \) does not meet \( \partial H^\infty \).

**Theorem 4.8.** — Let \( E \) be a set in \( M(H^\infty) \setminus \mathbb{D} \) that does not meet the Shilov boundary of \( H^\infty \). Then \( E \) is \( t \)-analytic for \( H^\infty \) if and only if \( E \) is \( t \)-analytic for \( H^\infty + C \).

**Proof.** — If \( E \) is \( t \)-analytic for \( H^\infty + C \), then it is easily seen that \( E \) is \( t \)-analytic for \( H^\infty \). Indeed, it suffices to observe that any open set \( U \) in \( M(H^\infty) \) induces the open set \( U \cap M(H^\infty + C) \) in \( M(H^\infty + C) \).

Conversely, let \( E \subseteq M(H^\infty + C) \) be \( t \)-analytic for \( H^\infty \) with \( E \cap \partial H^\infty = \emptyset \). Let \( f \in H^\infty + C \) vanish identically on \( E \cap \Omega \) for an open set \( \Omega \subseteq M(H^\infty + C) \) with \( E \cap \Omega \neq \emptyset \). Let \( x \in E \). Since \( E \cap \partial H^\infty = \emptyset \) we may use Lemma 4.6 to conclude that \( E \subseteq k(x) \). Moreover, by Lemma 4.4, \( k(x) \) is contained in a single fiber \( M_\lambda \). Now on fibers, \( (H^\infty + C)|_{M_\lambda} = H^\infty|_{M_\lambda} \). Thus we may choose \( F \in H^\infty \) such that \( F = f \) on \( M_\lambda \). Now for any open set \( W \) in \( M(H^\infty) \) with \( W \cap M(H^\infty + C) = \Omega \), we have \( F \equiv 0 \) on \( W \cap E \). Since \( E \) is \( t \)-analytic for \( H^\infty \), \( F \equiv 0 \) on \( E \) and so does \( f \). Hence \( E \) is \( t \)-analytic for \( H^\infty + C \).
Recall that a point \( x \in E \subseteq X \), \( X \) a topological space, is said to be an isolated point (for \( E \)), if there exists an open neighborhood \( U \) of \( x \) such that \( U \cap E = \{x\} \).

It follows as a special case of [2, Corollary 4.11] that if \( E \) is a \( t \)-analytic set for \( A \), then \( E \) either is a singleton or does not contain an isolated point.

**Theorem 4.9.** — Let \( E \) be a subset of \( \mathbb{D} \). Suppose that \( E \) contains more than one point. Then the following assertions are equivalent:

1. \( E \) is a strong uniqueness set for \( H^\infty \);
2. \( E \) is \( t \)-analytic for \( H^\infty \);
3. \( E \) does not contain any isolated point.

**Proof.** — (1) \( \implies \) (2) trivial.
(2) \( \implies \) (3) Suppose to the contrary that \( z_0 \in E \) is an isolated point. The function \( z - z_0 \) then vanishes in a relative open neighborhood of \( E \), but not at any other point. Thus \( E \) is no longer a \( t \)-analytic set.
(3) \( \implies \) (1) This follows immediately from the fact that the zeros of non-constant holomorphic functions are discrete (in \( \mathbb{D} \)). \( \square \)

**Theorem 4.10.** — Let \( E \) be a \( t \)-analytic set for \( H^\infty \) such that \( E \cap \mathbb{D} \neq \emptyset \).
Then
\[
E \subseteq \partial H^\infty \cup \overline{E \cap \mathbb{D}}.
\]

**Proof.** — Suppose contrariwise that there is some \( x \in E \setminus \partial H^\infty \) and \( x \notin \overline{E \cap \mathbb{D}} \).
Let \( z_0 \in E \cap \mathbb{D} \). Choose, as in Lemma 4.6, a Blaschke product \( B \) such that \( B \) vanishes identically on a neighborhood \( U^* \) of \( x \) in \( M(H^\infty + C) \). Let the open subset \( U \) of \( U^* \) satisfy \( x \in U \) and \( U \cap \overline{E \cap \mathbb{D}} = \emptyset \). We may also assume that \( B(z_0) \neq 0 \) (otherwise we just delete the zero \( z_0 \)). Let \( V \subseteq M(H^\infty) \) be open with \( V \cap M(H^\infty + C) = U \), \( z_0 \notin V \) and \( V \cap \overline{E \cap \mathbb{D}} = \emptyset \). Note that
\[
E \setminus (\overline{E \cap \mathbb{D}}) \subseteq M(H^\infty + C).
\]
Then
\[
V \cap E = V \cap (E \setminus \overline{E \cap \mathbb{D}}) = V \cap (E \setminus \overline{E \cap \mathbb{D}}) \cap M(H^\infty + C)
= U \cap (E \setminus \overline{E \cap \mathbb{D}}) = (U \cap E) \setminus \overline{E \cap \mathbb{D}} = U \cap E.
\]
Hence \( B \equiv 0 \) on \( V \cap E \), but \( B(z_0) \neq 0 \). Accordingly, \( E \) is not \( t \)-analytic. \( \square \)

As an immediate Corollary we have the following corona-type theorem.

**Corollary 4.11.** — Let \( E \) be a closed \( t \)-analytic set for \( H^\infty \) with \( E \cap \mathbb{D} \neq \emptyset \) and \( E \cap \partial H^\infty = \emptyset \). Then \( E = \overline{E \cap \mathbb{D}} \).

Let us note that the set of non-closed \( t \)-analytic sets for \( H^\infty \) is very huge. For example, in view of Corollary 2.2 one has that for all \( S \subseteq M(H^\infty + C) \) the set \( \mathbb{D} \cup S \) is a strong uniqueness set for \( H^\infty \).
Theorem 4.12. — Let $E \subseteq M(H^\infty)$ and suppose that $E \cap \mathbb{D} \neq \emptyset$ or $E \subseteq \partial H^\infty$. Then the following assertions are equivalent:

1. $E$ is a strong uniqueness set for $H^\infty$;
2. $E$ is $t$-analytic for $H^\infty$ and contains more than one point.

Proof. — (1) $\implies$ (2) is trivial. By Proposition 4.1, (2) $\implies$ (1) whenever $E \subseteq \partial H^\infty$. Now suppose that $E \cap \mathbb{D} \neq \emptyset$ and that $E$ contains more than one point. As previously mentioned, the $t$-analyticity of $E$ implies that $E \cap \mathbb{D}$ does not contain any isolated point. Hence, by Theorem 4.9, $E \cap \mathbb{D}$ is a strong uniqueness set. By Proposition 2.1, this implies that $E \cap \mathbb{D}$ is in $\mathcal{U}$, too.

Consider now the set $S := E \setminus E \cap \mathbb{D}$.

If $S = \emptyset$, then $E \subseteq E \cap \mathbb{D}$. Hence $E \cap \mathbb{D} \subseteq E \subseteq E \cap \mathbb{D}$. Since $E \cap \mathbb{D} \in \mathcal{U}$, we have, by Corollary 2.2, that $E \in \mathcal{U}$.

Let us now assume that $S \neq \emptyset$. By Theorem 4.10, $S \subseteq \partial H^\infty$. Note that $S$ is not a singleton, since otherwise $E$ would contain an isolated point. This would contradict the fact that $E$ is $t$-analytic.

Let $U$ be an open set in $M(H^\infty)$ with $U \cap S \neq \emptyset$ and let $f \in H^\infty$ be such that $f \equiv 0$ on $U \cap S$. By passing to a subset, we may assume that $U \cap E \cap \mathbb{D} = \emptyset$, but still $U \cap S \neq \emptyset$. Hence $U \cap S = U \cap E$. Since $E$ is $t$-analytic, we get that $f \equiv 0$ on $E$. In particular, $f \equiv 0$ on $S$. Thus $S$ is $t$-analytic. By Theorem 4.1, $S$ is in $\mathcal{U}$. By Proposition 2.7, $S \cup E \cap \mathbb{D} \in \mathcal{U}$. Since $E = S \cup E \cap \mathbb{D}$, we conclude that $E \in \mathcal{U}$. \hfill $\square$

Let $\mathcal{F}$ denote the class of subsets $F$ of $\mathbb{D}$ that do not contain any isolated points, let $\mathcal{U}_c$ denote the class of those strong uniqueness sets for $H^\infty$ that are closed. The following concluding theorems sum up the different situations dealt with above.

Theorem 4.13. — Let $E \subseteq M(H^\infty)$ be closed. Then $E \in \mathcal{U}_c\cup\{\emptyset\}$ if and only if $E = K \cup \mathcal{F}$, where $F \in \mathcal{F}$ and where $K \subseteq \partial H^\infty$ is a closed set such that for every Lebesgue measurable set $S \subseteq \mathbb{T}$ with strictly positive Lebesgue measure either $\hat{\sigma}(K \cap \{\hat{\chi}_S = 1\}) > 0$ or $K \cap \{\hat{\chi}_S = 1\} = \emptyset$.

Proof. — Let $E = K \cup \mathcal{F}$, where $K$ and $F$ satisfy the conditions above. By Theorem 4.3, $K \in \mathcal{U}$ whenever $K \neq \emptyset$. By Theorem 4.9, $F \in \mathcal{U}$ whenever $F \neq \emptyset$. By Proposition 2.7, $E \in \mathcal{U}$.

Conversely, let $E \in \mathcal{U}_c$. We discuss two cases:

Case 1. $E \cap \mathbb{D} \neq \emptyset$. Since strong uniqueness sets do not contain isolated points, $E \cap \mathbb{D} \in \mathcal{F}$. Hence, by Theorem 4.9, $E \cap \mathbb{D} \in \mathcal{U}$. Moreover, by Proposition 2.1, $E \cap \mathbb{D} \in \mathcal{U}$. If $E \cap \mathbb{D} = E$, then we are done. So suppose...
that \( E \cap \mathbb{D} \subsetneq E \). By the observation 2.3, \( E \setminus E \cap \mathbb{D} \in \mathcal{V} \), and so again,
\[
K := E \setminus \overline{E \cap \mathbb{D}} \in \mathcal{V}.
\]
But by Theorem 4.10, \( K \subseteq \partial H^\infty \). Hence we can conclude from Theorem 4.3 that \( K \) has the desired property. Since \( E = \overline{F} \cup K \), where \( F = E \cap \mathbb{D} \), we are done.

Case 2. \( E \cap \mathbb{D} = \emptyset \). Theorem 4.7 implies that either \( E \subseteq \partial H^\infty \), or \( E \cap \partial H^\infty = \emptyset \). We shall see that the hypotheses \( E \in \mathcal{Y} \) implies that the second case does not occur. So suppose that \( E \subseteq M(H^\infty + C) \setminus \partial H^\infty \). Then, by Lemma 4.6, \( E \subseteq k(x) \) for \( x \in E \). Hence the hull-kernel closure in \( M(H^\infty) \) of \( E \) is contained in \( k(x) \), too (see Lemma 4.5). Thus \( E \) cannot be a strong uniqueness set. We conclude that \( E \subseteq \partial H^\infty \). Using Theorem 4.3 again, we see that \( K := E \) has the property we wish. \( \square \)

Combining Theorems 4.3, 4.12 and 4.13, we get the following result.

**Theorem 4.14.** — The class \( \mathcal{A}_D \) of closed \( t \)-analytic sets for \( H^\infty \) that meet \( \mathbb{D} \) is given by \( \mathcal{A}_D = \mathcal{A}_1 \cup \mathcal{A}_2 \), where
\[
\mathcal{A}_1 = \{ \{ z_0 \} : z_0 \in \mathbb{D} \},
\]
and
\[
\mathcal{A}_2 = \{ K \cup \overline{F} : \emptyset \neq F \subset \mathbb{D}, F \in \mathcal{F}, K \subseteq \partial H^\infty, K \in \mathcal{Y}_c \text{ or } K = \emptyset \}.
\]

Finally, a combination of theorems 4.7, 4.8 and 4.12 yields:

**Theorem 4.15.** — The class \( \mathcal{A}_{\text{cor}} \) of closed \( t \)-analytic sets for \( H^\infty \) contained in the corona \( M(H^\infty + C) \) of \( H^\infty \) is given by \( \mathcal{A}_{\text{cor}} = \mathcal{A}_3 \cup \mathcal{A}_4 \), where
\[
\mathcal{A}_3 = \{ E : E \text{ \( t \)-analytic for } H^\infty + C \}
\]
and
\[
\mathcal{A}_4 = \{ E : E \subseteq \partial H^\infty, E \in \mathcal{Y}_c \}.
\]

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References


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