STABLE RANKS FOR THE REAL FUNCTION ALGEBRA $C(X, \tau)$

by

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Abstract. — We determine the Bass and topological stable ranks of the real algebra $C(X, \tau)$ of all complex-valued continuous functions on the compact Hausdorff space $X$ that satisfy $f \circ \tau = f$, where $\tau$ is a topological involution on $X$.

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1. Introduction

We solve a problem that apparently goes back to the seventies in the last century when the first investigations concerning stable ranks of function algebras were undertaken by L.N. Vasersthein. Let $C(X) = C(X, \mathbb{C})$ be the Banach algebra of all complex-valued continuous functions on the compact Hausdorff space $X$ endowed with the supremum-norm. Similarly, $C(X, \mathbb{R})$ is the algebra of all real-valued continuous functions on $X$. We are considering here the more general object

$$C(X, \tau) = \{ f \in C(X) : f \circ \tau = f \},$$

where $\tau$ is a topological involution on $X$. This algebra, that first appeared in a paper by Arens and Kaplansky [1], is meanwhile considered as a standard model for real function algebras; see the monograph [4] by Kulkarni and Limaye; in particular pages 2 and 28-29. It is our aim to calculate the Bass and topological stable ranks of $C(X, \tau)$, thus extending the classical results by Vasersthein [14] for the algebras $C(X)$ and $C(X, \mathbb{R})$. At the same time, our result will yield a unified approach to the determination of the stable ranks for several algebras of real-symmetric functions defined on planar sets or on compacta in $\mathbb{C}^n$ that appeared in [5, 6, 7].

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2. The terminology

Let $A$ be a commutative real or complex Banach algebra with unit element 1. An $n$-tuple $(f_1, \ldots, f_n) \in A^n$ is said to be invertible (or unimodular), if there exists $(x_1, \ldots, x_n) \in A^n$ such that $\sum_{j=1}^n x_j f_j = 1$. The set of all invertible $n$-tuples is denoted by $U_n(A)$. An $(n+1)$-tuple $(f_1, \ldots, f_n, g) \in U_{n+1}(A)$ is called reducible if there exists $(a_1, \ldots, a_n) \in A^n$ such that $(f_1 + a_1 g, \ldots, f_n + a_n g) \in U_n(A)$.

The Bass stable rank of $A$, denoted by $\text{bsr} A$, is the smallest integer $n$ such that every element in $U_{n+1}(A)$ is reducible. If no such $n$ exists, then $\text{bsr} A = \infty$.

The topological stable rank, $\text{tsr} A$, of $A$ is the least integer $n$ for which $U_n(A)$ is dense in $A^n$, or infinite if no such $n$ exists.

Note that if $\text{bsr} A = n$, then any $m + 1$-tuple $(f_1, \ldots, f_m, g)$ in $A$ with $m \geq n$ is reducible. Moreover, we always have $\text{bsr}(A) \leq \text{tsr} A$ (see [11] or [8, Theorem 2.7]).

Next, let us recall the definition of the notion of covering dimension (or Čech-Lebesgue dimension) as given in [2, p. 54] or [10, p. 111] that is intrinsically related to the concept of stable ranks. Let $Y$ be a normal topological space. The order of a family $\mathcal{A}$ of subsets of $Y$ is the largest integer $n$ such that $\mathcal{A}$ contains $n+1$ sets with a non-empty intersection. If no such integer exists, then $\mathcal{A}$ is said to have infinite order. The space $Y$ is said to have dimension $n$, denoted by $\dim Y = n$, if $n$ is the smallest integer such that every finite open cover of $Y$ has a finite open refinement of order $n$. If no such $n$ exists, then $\dim Y = \infty$. If $Y = \emptyset$, then we let $\dim Y = -1$.

Throughout this paper, $X$ will be a compact Hausdorff space and $\tau$ a topological involution on $X$; that is a homeomorphic selfmap of $X$ with $\tau \circ \tau$ being the identity. The set of fixed points of $\tau$ will be denoted by $E$. Note that $E$ is closed, but $E$ may be empty.

A subset $S$ of $X$ is called $\tau$-invariant, if $\tau(S) = S$. For $f \in C(X)$, let $\sigma(f) = f \circ \tau$. Then $\sigma$ is an involution on $C(X)$.

As usual, $Z(f) = \{x \in X : f(x) = 0\}$ denotes the zero set of $f \in C(X)$. The tuple $(f_1, \ldots, f_n) \in C(X)^n$ will sometimes be denoted by $f$ and $|f| := \sum_{j=1}^n |f_j|$ is the $\ell^1$-norm of $f$.

3. Extension of invertible tuples

The following result is in [4, p. 38].

**Proposition 3.1.** — The maximal ideal space (or set of nonzero multiplicative $\mathbb{C}$-linear functionals on $C(X, \tau)$) can be identified (via point evaluations)
with $X$ itself. (Here $C$ is regarded as a vector space over $\mathbb{R}$.) In particular, $f \in C(X, \tau)$ is invertible if and only if $f$ has no zeros in $C(X, \tau)$.

Let $(f_1, \ldots, f_n) \in C(X, \tau)^n$. Then the Bézout equation $\sum_{j=1}^n q_j f_j = 1$ has a solution $(q_1, \ldots, q_n) \in C(X, \tau)^n$ if and only if $\sum_{j=1}^n |f_j| \geq \delta > 0$, or equivalently, if and only if $\bigcap_{j=1}^n Z(f_j) = \emptyset$. If $(Q_1, \ldots, Q_n) \in C(X)^n$ satisfies $\sum_{j=1}^n Q_j f_j = 1$, then $q_j := \frac{1}{2}(Q_j + \sigma(Q_j))$ satisfies the Bézout equation in $C(X, \tau)$.

We need the following Tietze-Urysohn type Lemma.

**Lemma 3.2.** — 1) Let $S$ be a closed $\tau$-invariant subset of $X$ and $f \in C(S, \tau)$. Then $f$ admits an extension to a function $F$ in $C(X, \tau)$. (We call $F$ a $\tau$-invariant Tietze extension of $f$).

2) Let $X_0$ and $X_1$ be two disjoint, closed $\tau$-invariant subsets of $X$. Then there exists $f \in C(X, \tau)$ such that $f \equiv 0$ on $X_0$, $f \equiv 1$ on $X_1$ and $0 \leq f \leq 1$. If, additionally, $X_0$ and $X_1$ are $G_\delta$ sets, then $f$ can be chosen so that $X_0 = f^{-1}\{0\}$ and $X_1 = f^{-1}\{1\}$.

**Proof.** — 1) Let $f^* \in C(X)$ be any Tietze extension of $f$. Then

$$F = \frac{f^* + \sigma(f^*)}{2} \in C(X, \tau)$$

is the desired extension of $f$.

2) Let $f^*$ be a Urysohn map; that is $f^* \in C(X)$, $0 \leq f^* \leq 1$, $f^* \equiv 0$ on $X_0$ and $f^* \equiv 1$ on $X_1$. Then $f = (f^* + \sigma(f^*))/2$ does the job. If $X_0 = \bigcap_{j=1}^\infty G_j$, where $G_j$ are open subets of $X$ with $\overline{G_j} \cap X_1 = \emptyset$, then we choose $f_j \in C(X, \tau)$ such that $f_j \equiv 0$ on $X_0$, $f_j \equiv 1$ on $X \setminus G_j$ and $0 \leq f_j \leq 1$. Then

$$F_0 := \sum_{j=1}^\infty \frac{f_j}{2} \in C(X, \tau),$$

vanishes exactly on $X_0$, is constant 1 on $X_1$ and $0 \leq F_0 \leq 1$.

Now let $F_1 \in C(X, \tau)$ vanish exactly on $X_1$ and suppose that $0 \leq F_1 \leq 1/2$. Let $f := F_0(1 - F_1)$. Then $f \in C(X, \tau)$, $0 \leq f \leq 1$, $f^{-1}\{0\} = X_0$ and $f^{-1}\{1\} = X_1$. 

**Corollary 3.3.** — Let $E$ be the set of fixed points of $\tau$ on $X$. Then $C(X, \tau)|_E = C(E, \mathbb{R})$.

**Proof.** — If $x \in E$, then $f \in C(X, \tau)$ implies that

$$f(x) = f(\tau(x)) = (f \circ \tau)(x) = \overline{f(x)}.$$
Hence $C(X, \tau)|_E \subseteq C(E, \mathbb{R})$. If, on the other hand, $g \in C(E, \mathbb{R})$, then the $\tau$-invariance of $E$ and Lemma 3.2 (1) imply that $g$ admits an extension to a function $f \in C(X, \tau)$. \hfill \Box

Let $A$ denote either $C(X), C(X, \mathbb{R})$ or $C(X, \tau)$. The following theorem is well known for $C(X)$ and $C(X, \mathbb{R})$. However, by exactly the same reasoning as in [12], one also can prove it for $C(X, \tau)$.

**Theorem 3.4.** — Let $(f_1, \ldots, f_n, g)$ be an invertible $(n+1)$-tuple in $A$. Suppose that $g$ is not invertible. Then the following assertions are equivalent:

1. $(f_1, \ldots, f_n, g)$ is reducible;
2. $(f_1, \ldots, f_n)|_{Z(g)}$ admits an extension to an invertible $n$-tuple $(F_1, \ldots, F_n) \in U_n(A)$.

We want to present the following variant, that suffices for our purposes, but that is much simpler to prove than Theorem 3.4.

**Theorem 3.5.** — Let $(f_1, \ldots, f_n, g)$ be an invertible $(n+1)$-tuple in $C(X, \tau)$. Suppose that $g$ is not invertible. Then the following assertions are equivalent:

1. $(f_1, \ldots, f_n, g)$ is reducible;
2. There exists a $\tau$-invariant neighborhood $V$ of $Z(g)$ such that $(f_1, \ldots, f_n)|_V$ admits an extension to an invertible $n$-tuple $(F_1, \ldots, F_n) \in U_n(C(X, \tau))$.

**Proof.** — (1) $\implies$ (2) We may assume without loss of generality that $(f_1, \ldots, f_n)$ is not invertible. Hence $g \neq 0$. By our hypothesis, $(F_1^*, \ldots, F_n^*) := (f_1 + x_1 g, \ldots, f_n + x_n g) \in U_n(C(X, \tau))$ for some $(x_1, \ldots, x_n) \in C(X, \tau)^n$. Note that not all functions $x_j$ can vanish identically on $X$, since otherwise $(f_1, \ldots, f_n)$ would be invertible. Suppose that $\sum_{j=1}^n |F_j^*| \geq \delta > 0$ on $X$. Let

$$V = \{ x \in X : |g(x)| < \frac{\delta}{4\sum_{j=1}^n \|x_j\|_{\infty}} \}$$

and

$$W = \{ x \in X : |g(x)| \geq \frac{\delta}{2\sum_{j=1}^n \|x_j\|_{\infty}} \}.$$

Of course, both sets are $\tau$-invariant. Since $Z(g) \neq \emptyset$ and $Z(g) \subseteq V$, we see that $V \neq \emptyset$. Moreover, by taking $\delta$ sufficiently small, $W \neq \emptyset$. Next, choose $G \in C(X, \tau)$ so that $0 \leq G \leq 1$, $G \equiv 0$ on $V$ and $G \equiv 1$ on $W$. Let

$$F_j = f_j + x_j g G.$$

We claim that $(F_1, \ldots, F_n) \in U_n(C(X, \tau))$. In fact, by distinguishing the cases $x \in W$ and $x \notin W$, the formula $F_j = F_j^* + x_j g (G - 1)$ implies that

$$\sum_{j=1}^n |F_j| \geq \sum_{j=1}^n |F_j^*| - \sum_{j=1}^n |x_j g (G - 1)| \geq \delta - \delta/2 = \delta/2.$$
Hence the $n$-tuple $(F_1, \ldots, F_n)$ is invertible. Since $(F_1, \ldots, F_n)$ obviously extends $(f_1, \ldots, f_n)|_V$ we are done.

It remains to prove (2) $\Rightarrow$ (1). Let $(F_1, \ldots, F_n)$ be an invertible extension of $(f_1, \ldots, f_n)|_V$ for some $\tau$-invariant open neighborhood $V$ of $Z(g)$. Then

$$(x_1, \ldots, x_n):=((F_1-f_1)/g, \ldots, (F_n-f_n)/g)$$

is well defined on $X$ and so

$$(F_1, \ldots, F_n) = (f_1+x_1g, \ldots, f_n+x_ng)$$

is the desired reduction in $C(X, \tau)$ of the $(n+1)$-tuple $(f_1, \ldots, f_n, g)$. \hfill \Box

**Theorem 3.6.** — Let $X$ be compact, $S \subseteq X$ closed, and let $A$ be either $C(X)$ or $C(X, \mathbb{R})$. Suppose that $\text{bsr} \ A \leq N$. Then every invertible $N$-tuple $(f_1, \ldots, f_N) \in C(S)^N$, respectively $C(S, \mathbb{R})^N$, has an extension to an invertible $N$-tuple in $C(X)$, respectively $C(X, \mathbb{R})$.

If $A = C(X, \tau)$ and $S$ is $\tau$-invariant, then the assertion holds for $C(X, \tau)$ and $C(S, \tau)$ as well.

**Proof.** — Let $(f_1^*, \ldots, f_N^*) \in A^n$ be a Tietze extension of $(f_1, \ldots, f_N)$ from $S$ to $X$. Let $g \in A$ satisfy $g \equiv 0$ on $S$ and $g \equiv 1$ on $\bigcap_{j=1}^N Z(f_j^*)$. Then $(f_1^*, \ldots, f_N^*, g)$ is an invertible $(N+1)$-tuple in $A$. Since $\text{bsr} \ A \leq N$, $(f_1^*, \ldots, f_N^*, g)$ is reducible. Hence, by Theorem 3.5, there exists a $\tau$-invariant neighborhood $V$ of $Z(g)$ so that $(f_1^*, \ldots, f_N^*)|_V$ admits an extension to an invertible $N$-tuple in $A$. Hence $(f_1^*, \ldots, f_N^*)|_{Z(g)}$ and so $f := (f_1^*, \ldots, f_N^*)|_{Z(g) \cap S}$ do. Since $Z(g) \cap S = S$, we conclude that $f = (f_1, \ldots, f_N)$. Thus $(f_1, \ldots, f_N)$ admits the desired extension. \hfill \Box

### 4. Maximal open asymmetric sets

Asymmetric sets (these are sets $S \subseteq X$ satisfying $S \cap \tau(S) = \emptyset$) will play a central role in the proof of our main theorem. They will allow us to construct the invertible extensions of tuples in $C(X, \tau)$ discussed in the previous section. This section will be devoted to these asymmetric sets. Examples will clarify the great variety in which they occur.

In the following, $\overline{M}$ denotes the closure of $M \subseteq X$, $M^\circ$ the interior of $M$ and $\partial M$ the topological boundary of $M$.

**Lemma 4.1.** —

1. Let $U \subseteq X$ be open. Then $(U)^\circ = U \iff \partial U = \partial U$;
2. Let $U$ and $V$ be two disjoint open subsets of $X$ satisfying $(U)^\circ = U$ and $(V)^\circ = V$. Further, let $E \subseteq X$ be closed and disjoint with $U$ and $V$ and suppose that $X = U \cup E \cup V$. Then $\partial U \setminus E = \partial V \setminus E$.
Proof. — (1) If \((U)^{\circ} = U\), then \(\partial U = U \setminus (U)^{\circ} = U \setminus U = \emptyset\). On the other hand, if \(\partial U = \partial U\), then \(U \subseteq (U)^{\circ} \subseteq U \setminus \partial U = U \setminus \partial U = U\).

(2) Let \(x \in \partial U \setminus E = \partial U \setminus E\). Then, by approaching \(x\) with a net of points outside \(U\) it is readily seen that \(x \in \overline{V} \setminus V = \partial V\). Interchanging the role of \(U\) and \(V\) yields the assertion. \hfill \Box

**Theorem 4.2.** — Let \(E = \{x \in X : \tau(x) = x\}\) be the set of fixed points of \(\tau\). Then the following holds:

1. Each \(x \in X \setminus E\) is contained in a maximal open set \(U = U_{\text{max}}\) with \(U \cap \tau(U) = \emptyset\).
2. \(X = \overline{U_{\text{max}}} \cup E \cup \tau(U_{\text{max}})\).
3. \((U_{\text{max}})^{\circ} = U_{\text{max}}\) and \(\partial U_{\text{max}} = \partial \overline{U_{\text{max}}}\).
4. The boundary \(\partial U_{\text{max}}\) of \(U_{\text{max}}\) is \(\tau\)-invariant.

\(U_{\text{max}}\) is a called a maximal open asymmetric set.

Proof. — (1) Let \(x \in X \setminus E\). Since \(\tau(x) \neq x\), the normality of \(X\) implies that there exist two open sets \(U(x)\) and \(U(\tau(x))\) containing \(x\), respectively \(\tau(x)\), such that \(\overline{U(x)} \cap \overline{U(\tau(x))} = \emptyset\). Since \(\tau\) is continuous, there is an open neighborhood \(V \subseteq U(x)\) of \(x\) such that \(\tau(V) \subseteq U(\tau(x))\). Thus \(V \cap \tau(V) = \emptyset\).

Now let \(\mathcal{S} = \{W : x \in W, \ W \text{open}, \ W \cap \tau(W) = \emptyset\}\) be partially ordered by set inclusion. Since \(V \in \mathcal{S}\), we have \(\mathcal{S} \neq \emptyset\). We claim that in \(\mathcal{S}\) each chain \(\{W_{\lambda} : \lambda \in \Lambda\}\), \(\Lambda\) a totally ordered set of indices, has an upper bound \(\Omega\). In fact, let \(\Omega = \bigcup_{\lambda \in \Lambda} W_{\lambda}\). Then \(\Omega\) is open and \(\Omega \cap \tau(\Omega) = \emptyset\).

To see this latter point, fix \(\lambda_0 \in \Lambda\). Then

\[
W_{\lambda_0} \cap \tau(\Omega) = \bigcup_{\lambda}(\tau(W_{\lambda}) \cap W_{\lambda_0})
\]

\[
= \bigcup_{\lambda \leq \lambda_0} (\tau(W_{\lambda}) \cap W_{\lambda_0}) \cup \bigcup_{\lambda > \lambda_0} (\tau(W_{\lambda}) \cap W_{\lambda_0})
\]

\[
\subseteq \bigcup_{\lambda \leq \lambda_0} (\tau(W_{\lambda}) \cap W_{\lambda_0}) \cup \bigcup_{\lambda > \lambda_0} (\tau(W_{\lambda}) \cap W_{\lambda}) = \emptyset.
\]

Zorn’s Lemma now yields the desired maximal set \(U_{\text{max}}\).

(2) Let \(y \in X \setminus E\) and suppose that \(y \notin \overline{U_{\text{max}}} \cup \overline{\tau(U_{\text{max}})}\). As in the paragraph above, since \(y \neq \tau(y)\), there exists an open neighborhood \(V\) of \(y\) such that \(V \cap \tau(V) = \emptyset\). We may choose additionally \(V\) so that \(V \cap U_{\text{max}} = \emptyset\) and \(V \cap \tau(U_{\text{max}}) = \emptyset\). Let \(W := U_{\text{max}} \cup V\). Then \(W \cap \tau(W) = \emptyset\). But \(x \in U_{\text{max}} \subseteq W\), contradicting the maximality of \(U_{\text{max}}\). Thus \(y \in \overline{U_{\text{max}}} \cup \tau(U_{\text{max}})\).

Altogether, we have shown that \(X = \overline{U_{\text{max}}} \cup \overline{U_{\text{max}}} \cup E\).
(3) In view of Lemma 4.1 it suffices to show that $(U_{\text{max}})^\circ = U_{\text{max}}$. Obviously $U_{\text{max}} \subseteq (U_{\text{max}})^\circ$. Moreover, $(U_{\text{max}})^\circ \cap \tau(U_{\text{max}}) = \emptyset$ (otherwise $\tau(U_{\text{max}})$, an open set, would meet $(U_{\text{max}})^\circ$ and so $\tau(U_{\text{max}}) \cap U_{\text{max}} \neq \emptyset$. Hence $\tau(U_{\text{max}}) \cap U_{\text{max}} \neq \emptyset$, a contradiction. Now

$$\emptyset = (U_{\text{max}})^\circ \cap \tau(U_{\text{max}}) = (U_{\text{max}})^\circ \cap \tau(U_{\text{max}}) \supseteq (U_{\text{max}})^\circ \cap \tau((U_{\text{max}})^\circ).$$

Thus $(U_{\text{max}})^\circ$ is asymmetric. Note that $U_{\text{max}} \subseteq (U_{\text{max}})^\circ$. The maximality of $U_{\text{max}}$ now implies that $U_{\text{max}} = (U_{\text{max}})^\circ$.

(4) Follows from (3) and Lemma 4.1 by using the facts that

$$\partial(\tau(U_{\text{max}})) = \tau(\partial U_{\text{max}}) \quad \text{and} \quad \tau\left((U_{\text{max}})^\circ\right) = \tau(U_{\text{max}})^\circ.$$

\[\Box\]

**Example 4.3.** — There exists $(X, \tau)$ for which

i) $E = \emptyset$ and $\partial U_{\text{max}} = \emptyset$ or ii) $E = \emptyset$ and $\partial U_{\text{max}} \neq \emptyset$ or iii) $E = \partial U_{\text{max}} \neq \emptyset$ or iv) $\emptyset \neq E \subsetneq \partial U_{\text{max}}$ or v) $E^\circ = \emptyset$ and $\partial U_{\text{max}} = \emptyset$ or vi) $\partial U_{\text{max}} = \partial(E^\circ) \neq \emptyset$.

**Proof.** — i) Let $X = \{|z| \leq 1\} \cup \{|z - 3| \leq 1\}$ and $\tau(z) = z + 3$ if $|z| \leq 1$ and $\tau(z) = z - 3$ if $|z - 3| \leq 1$. Then there are exactly two maximal open asymmetric sets, namely $\{|z| \leq 1\}$ and $\{|z - 3| \leq 1\}$, and no fixed points.

ii) Let $X = \{1 \leq |z| \leq 2\}$ and $\tau(z) = -z$. Then $\tau$ has no fixed points. Each point $p \in X$ belongs to infinitely many maximal open asymmetric sets: just take the intersection of $X$ with any half-plane passing through 0 and containing $p$ (see figure 1).

![Figure 1. Some maximal domains](image)
iii) Let \( X = \{|z| \leq 1 \} \) and \( \tau(z) = \overline{z} \). Then there are exactly two maximal open asymmetric sets: \( \{ z \in X : \text{Im } z > 0 \} \) and \( \{ z \in X : \text{Im } z < 0 \} \). Moreover, the boundary of these sets coincides with the set \([-1, 1]\) of fixed points of \( \tau \).

iv) Let \( X = \{|z| \leq 1 \} \) and \( \tau(z) = \frac{a - z}{1 - \overline{a}z} \), where \( |a| < 1 \). Then \( \tau \) is an elliptic automorphism of \( X \) and coincides with its own inverse. Each point in \( X \) different from the unique fixed point \( p \) of \( \tau \) in \( X \) belongs to infinitely many maximal open asymmetric sets \( U \). The boundary of each such sets coincides with \( X \cap C \), where \( C \) is a circle passing through \( p \) and orthogonal to the unit circle (see figure 2).

![Figure 2](image)

**Figure 2.** \( U_1 \) grey; \( U_2 \) hatched

v) Let \( X = \{|z + 3| \leq 1\} \cup \{|z| \leq 1\} \cup \{|z - 3| \leq 1\} \) and \( \tau(z) = z + 6 \) if \( |z + 3| \leq 1 \), \( \tau(z) = z - 6 \) if \( |z - 3| \leq 1 \) and \( \tau(z) = z \) if \( |z| \leq 1 \).

vi) Let \( X \) be the union of the closed unit disk with the boundary \( Q \) of the square \([-1, 1] \times [-2, 2]\) (see figure 3). For \( \tau \) we may take the function given by \( \tau(z) = \overline{z} \) if \( z \) belongs to \( Q \) and \( \tau(z) = z \) if \( |z| \leq 1 \). Hence \( \partial E = \{-1, 1\} \).

5. **The Bass and topological stable rank of \( C(X, \tau) \)**

First we recall Vaserstein’s result \([14]\) (see also \([7]\) for a self-contained, easy proof.)

**Theorem 5.1.** — Let \( X \) be a compact Hausdorff space. Then

\[
\text{tsr } C(X, \mathbb{C}) = \text{bsr } C(X, \mathbb{C}) = \left\lfloor \frac{\dim X}{2} \right\rfloor + 1
\]

\[
\text{tsr } C(X, \mathbb{R}) = \text{bsr } C(X, \mathbb{R}) = \dim X + 1.
\]
The proof of the following theorem works along the same lines as that we developed for the algebra $C(X)$ in [8]. For the reader’s convenience we present the details adapted to the new algebra here.

**Theorem 5.2.** — Let $X$ be a compact Hausdorff space. Then $\text{bsr} C(X, \tau) = \text{tsr} C(X, \tau)$.

**Proof.** — Since $\text{bsr} A \leq \text{tsr} A$ for any commutative unital Banach algebra $A$ (see [11]), it remains to prove the reverse inequality. Let $N = \text{bsr} C(X, \tau) < \infty$ and let $F := (f_1, \ldots, f_N)$ be an $N$-tuple in $C(X, \tau)$. If $F$ is invertible, we are done. So we may assume that the $f_j$ have at least one zero in common.

Consider the sets

$$E_n = \{ x \in X : \sum_{j=1}^N |f_j(x)| \geq 1/n \}.$$

Choose by Lemma 3.2 a function $g_n \in C(X, \tau)$ with $0 \leq g_n \leq 1$ such that $g_n$ vanishes identically on $E_n$ and is constant one on $\bigcap_{j=1}^N Z(f_j)$. Then the $(N+1)$-tuple $(f_1, \ldots, f_N, g_n)$ is invertible in $C(X, \tau)$. Since $\text{bsr} C(X, \tau) = N$, this tuple is reducible. Hence there exist $h_j \in C(X, \tau)$ so that $F_n := (f_1 + h_1 g_n, \ldots, f_N + h_N g_n)$ is invertible in $C(X, \tau)$. Distinguishing the three cases $x \in E_n$, $x \in X \setminus E_n$, but $F(x) \neq 0$, and $F(x) = 0$ readily yields that on $X$

$$\left| F - \left( |F| + \frac{1}{n} \right) \frac{F_n}{|F_n|} \right| \leq \frac{3}{n}.$$ 

Thus $\text{tsr} C(X, \tau) \leq \text{bsr} C(X, \tau)$. □

An important tool for the proof of our main theorem will be the following result from dimension theory.

**Theorem 5.3.** — [9, p. 54]

Let $X$ be a normal space and $E$ a closed subset of $X$. If $\dim E \leq n$ and $\dim S \leq n$ for any closed set $S \subseteq X$ with $E \cap S = \emptyset$, then $\dim X \leq n$. In particular, if $\dim X = \infty$ and $\dim E = n < \infty$ then, for every $m > n$, there is a closed set $S \subseteq X$, disjoint from $E$, with $\dim S \geq m$.

**Lemma 5.4.** — [4, Lemma 1.3.7, p. 30]

For every $x \in X \setminus E$ there exists $f_x \in C(X, \tau)$ with $f_x(x) = i$ and $f_x(\tau(x)) = -i$.

**Lemma 5.5.** — Let $S \subseteq X$ be a closed $\tau$-invariant subset of $X$ and $E$ the set of fixed points of $\tau$. Then, for every $x \in X \setminus (E \cup S)$, there exists $q_x \in C(X, \tau)$ with $q_x \equiv 0$ on $E \cup S$, $q_x(x) = i$ and $q_x(\tau(x)) = -i$. 
Proof. — According to Lemma 3.2, choose \( p_x \in C(X, \tau) \) with \( p_x \equiv 0 \) on \( E \cup S \) and \( p_x(x) = p_x(\tau(x)) = 1 \). Now we multiply \( p_x \) with the function \( f_x \) given by Lemma 5.4. The product \( q_x = p_x f_x \) then is the desired function.

Note that \( S \) may be the empty set in the Lemma above.

**Lemma 5.6.** — Let \( S \) be a closed \( \tau \)-invariant subset of \( X \). Then \( C(X, \tau)|_S = C(S, \tau) \) and

\[
\text{bsr } C(X, \tau) \geq \text{bsr } C(X, \tau)|_S.
\]

Proof. — Due to the invariance of \( S \) under \( \tau \), the assertion \( C(X, \tau)|_S = C(S, \tau) \) follows immediately from Lemma 3.2. Now we use Theorem 5.2 telling us that the Bass stable rank coincides with the topological stable rank for spaces of type \( C(X, \tau) \).

If \( \text{bsr } C(X, \tau) = \text{tsr } C(X, \tau) = \infty \), then there is nothing to prove. So, we may assume that \( N := \text{tsr } C(X, \tau) < \infty \). Let \( (f_1, \ldots, f_N) \) be an \( N \)-tuple in \( C(S, \tau) \). In virtue of Lemma 3.2, we may extend this tuple to an \( N \)-tuple \( (F_1, \ldots, F_N) \) in \( C(X, \tau) \). Since \( \text{tsr } C(X, \tau) = N \), we can approximate \( (F_1, \ldots, F_N) \) uniformly on \( X \) by an invertible \( N \)-tuple \( (H_1, \ldots, H_N) \) in \( C(X, \tau) \). The restriction of \( (H_1, \ldots, H_N) \) to \( S \) now yields the desired approximation of \( (f_1, \ldots, f_N) \). Thus \( \text{bsr } C(S, \tau) = \text{tsr } C(S, \tau) \leq N \).

**Lemma 5.7.** — Let \( R \subseteq X \) be closed and suppose that \( R \cap \tau(R) = \emptyset \). Then \( C(X, \tau)|_R = C(R) \) and

\[
\text{bsr } C(X, \tau) \geq \text{bsr } C(X, \tau)|_R.
\]

Proof. — Let \( q \in C(R) \). Consider the \( \tau \)-invariant set \( S = R \cup \tau(R) \). Let

\[
f(x) = \begin{cases} 
q(x) & \text{if } x \in R \\
q(\tau(x)) & \text{if } x \in \tau(R). 
\end{cases}
\]

Then \( f \) is well defined and so \( f \in C(S, \tau) \). By Lemma 3.2, \( f \) admits an extension to a function \( F \in C(X, \tau) \). Thus \( C(X, \tau)|_R = C(R) \). The assertion on the stable rank now follows as above in the proof of Lemma 5.6.

**Lemma 5.8.** — Suppose that the topological involution \( \tau \) has no fixed points on \( X \). Then \( X \) can be covered by finitely many closed sets \( S_j \) such that \( S_j \cap \tau(S_j) = \emptyset \) for every \( j \).

Proof. — For each \( x \in X \) let \( U_{\text{max}}(x) \) be a maximal open asymmetric set containing \( x \) (see Theorem 4.2). Recall that \( U_{\text{max}}(x) \cap \tau(U_{\text{max}}(x)) = \emptyset \). Let \( V(x) \) be an open set in \( X \) and \( C(x) \) a closed set satisfying

\[
x \in V(x) \subseteq C(x) \subseteq U_{\text{max}}(x), x \in X.
\]
Since $X$ is compact, there exist finitely many $V(x)$ covering $X$; say

$$X = \bigcup_{j=1}^{n} V(x_j).$$

Now the sets $S_j := C(x_j)$ do the job.

We are now able to prove the main result of this paper. This will confirm the conjecture given in [5].

**Theorem 5.9.** — Let $X$ be a compact Hausdorff space, and $\tau$ a topological involution of $X$. Denote the set of fixed points of $\tau$ by $E$. Then

$$\text{bsr} C(X, \tau) = \text{tsr} C(X, \tau) = \max \left\{ \left\lceil \frac{\dim X}{2} \right\rceil, \dim E \right\} + 1.$$

**Proof.** — By Theorem 5.2, it suffices to prove the assertion on the Bass stable rank. Let

$$N := \max \left\{ \left\lceil \frac{\dim X}{2} \right\rceil, \dim E \right\} + 1.$$

Note that $N = \infty$ is allowed.

**Step 1** We first show that $\text{bsr} C(X, \tau) \geq N$.

By Corollary 3.3, $C(X, \tau)|_E = C(E, \mathbb{R})$. Hence, by Lemma 5.6 and Vaserstein’s Theorem 5.1

$$\text{bsr} C(X, \tau) \geq \text{bsr} C(X, \tau)|_E = \text{bsr} C(E, \mathbb{R}) = \dim E + 1.$$

If $\dim E = \dim X$, then we are done, since $\left\lceil \frac{\dim X}{2} \right\rceil \leq \dim X$. So suppose that $\dim E < \dim X$. This implies that $\dim E$ is finite.

**Case 1** $\dim X < \infty$.

We show the existence of a closed set $R \subseteq X$ with $\dim R = \dim X$ and such that $R \cap \tau(R) = \emptyset$.

By Theorem 5.3, there exists a closed set $M \subseteq X$, $M \cap E = \emptyset$, for which $\dim M = \dim X$. Let $S = M \cup \tau(M)$. Then $S$ is $\tau$-invariant and $S \cap E = \emptyset$. Moreover, since $\tau$ is a homeomorphism, and since for normal spaces, the covering dimension is monotonic on closed sets (see [2, p. 209]), we have that $\dim S = \dim M = \dim X$. By Lemma 5.8, $S = \bigcup_{j=1}^{p} S_j$, where $S_j$ is closed and satisfies $S_j \cap \tau(S_j) = \emptyset$ for every $j$.

By the sum-theorem for the covering dimension, (see [2, p.42, Theorem 1.5.3]), at least one of the closed sets $S_j$ must have the dimension of $S$. Thus we have shown that there exists a closed set $R \subseteq X$ with $\dim R = \dim X$ and such that $R \cap \tau(R) = \emptyset$. In particular $R \cap E = \emptyset$. By Lemma 5.7 and Vaserstein’s theorem 5.1

$$\text{bsr} C(X, \tau) \geq \text{bsr} C(X, \tau)|_R = \text{bsr} C(R) = \left\lceil \frac{\dim R}{2} \right\rceil + 1 = \left\lceil \frac{\dim X}{2} \right\rceil + 1.$$

Altogether, we have shown that

$$\text{bsr} C(X, \tau) \geq \max \left\{ \left\lceil \frac{\dim X}{2} \right\rceil, \dim E \right\} + 1 = N.$$
Case 2 \( \dim X = \infty \).

Fix \( m > \dim E \). By Theorem 5.3, there exists a closed set \( M_m \subseteq X \), \( M_m \cap E = \emptyset \), for which \( \dim M_m \geq m \). Let \( S_m = M_m \cup \tau(M_m) \). Then \( S_m \cap E = \emptyset \) and \( \dim S_m = \dim M_m \). Note that \( \dim S_m = \infty \) is possible. Exactly as in the first case, one shows the existence of a closed set \( R_m \subseteq X \) with \( \dim R_m \geq m \) and such that \( R_m \cap \tau(R_m) = \emptyset \) (just replace “= \( \dim X \)” by “\( \geq m \)”). Then again,

\[
\bsr C(X, \tau) \geq \bsr C(X, \tau)|_{R_m} = \bsr C(R_m) = \left\lceil \frac{\dim R_m}{2} \right\rceil + 1 \geq \left\lceil \frac{m}{2} \right\rceil + 1.
\]

Since \( m \) was arbitrary, we conclude that \( \bsr C(X, \tau) = \infty \).

**Step 2** Here we show that \( \bsr C(X, \tau) \leq N \).

We may assume that \( N < \infty \). Let \( (f_1, \ldots, f_N, g) \) be an invertible \((N+1)\)-tuple in \( C(X, \tau) \). To show that \((f_1, \ldots, f_N, g)\) is reducible, it suffices to show, by Theorem 3.4, that \((f_1, \ldots, f_N)|_{Z(g)}\) admits an invertible extension to \( X \) whenever \( Z(g) \neq \emptyset \).

(Or, if we only want to use Theorem 3.5, it suffices to show that \((f_1, \ldots, f_N)|_{Z(g)\cap E}\) admits an invertible extension to \( X \), where \( G \in C(X, \tau) \) is chosen so that \( G \equiv 0 \) on \( U := \{x \in X : |g(x)| < \varepsilon \} \) and \( G \equiv 1 \) on \( \bigcap_{j=1}^N Z(f_j) \) for some \( \varepsilon \) that guarantees that \( \inf_{x \in \cup} \sum_{j=1}^N |f_j(x)| > 0 \).)

**Step 2.1** Let \( f = (f_1, \ldots, f_N) \) and \(|f| = \sum_{j=1}^N |f_j|\). We claim that \(|f|_{Z(g)}\) admits an extension to an invertible \( N \)-tuple \( f^{[1]} \) in \( C(E \cup Z(g), \tau) \) whenever \( E \) is nonvoid.

In fact, by Corollary 3.3, \( C(X, \tau)|_E = C(E, \tau) = C(E, \mathbb{R}) \). Now \( \bsr C(E, \mathbb{R}) = \dim E + 1 \leq N \). In particular \((f_1, \ldots, f_N, g)|_E\) is reducible in \( C(E, \mathbb{R}) \). Thus, by Theorem 3.4, \(|f|_{Z(g)\cap E}\) admits an extension to an invertible \( N \)-tuple \( h \) in \( C(E, \tau) \) whenever \( Z(g) \cap E \neq \emptyset \). Let \( h = (1, 0, \ldots, 0) \) if \( Z(g) \cap E = \emptyset \).

Now define \( f^{[1]} \) by

\[
f^{[1]}(x) = \begin{cases} h(x) & \text{if } x \in E \\ f(x) & \text{if } x \in Z(g). \end{cases}
\]

Since \( h = f \) on \( Z(g) \cap E \), the \( N \)-tuple \( f^{[1]} \) is well defined and so continuous on \( E \cup Z(g) \). Since \( E \) and \( Z(g) \) are \( \tau \)-invariant, we conclude that \( f^{[1]} \) is an invertible \( N \)-tuple in \( C(E \cup Z(g), \tau) \).

If \( E = \emptyset \), we let \( f^{[1]} := f \) on \( Z(g) \).

**Step 2.2** Let \( F^{[1]} \) be a \( \tau \)-invariant Tietze extension of \( f^{[1]} \in C(E \cup Z(g), \tau) \) to an \( N \)-tuple in \( C(X, \tau) \) (see Lemma 3.2). Let

\[
S = \{x \in X : |F^{[1]}(x)| = 0\},
\]
and consider the \(\tau\)-invariant closed neighborhood

\[ K = \{ x \in X : |F[x]| \leq \varepsilon \} \]

of \( S \), where \( \varepsilon > 0 \) is so small that

\[ (Z(g) \cup E) \cap K = \emptyset. \]

Let

\[ V = \{ x \in X : |F[x]| \geq \varepsilon \}. \]

Note that \( V \) is \( \tau \)-invariant and that \( Z(g) \cup E \subseteq V \).

We claim that \( F[1]|_V \) admits an extension to an invertible \( N \)-tuple \( F[2] \) in \( C(X, \tau) \) giving the desired extension of \( f|_{Z(g)} \).

Proof of that claim. Since \( K \) is \( \tau \)-invariant, and \( \tau \) has no fixed points in \( K \), we may use Lemma 5.8 to write

\[ K = \bigcup_{j=1}^n A_j, \]

where \( A_j \) are closed sets with \( A_j \cap \tau(A_j) = \emptyset \) for every \( j \). Since \( \tau \) is a homeomorphism and \( K \) is \( \tau \)-invariant, we also have that

\[ K = \bigcup_{j=1}^n \tau(A_j). \]

Now for each \( j \),

\[ \dim (V \cup A_j) \leq \dim X \]

and so

\[ \text{bsr} \ C(V \cup A_1) \leq \text{bsr} \ C(X) = \left[ \frac{\dim X}{2} \right] + 1 \leq N. \]

Thus, by Theorem 3.6 applied to the algebra \( C(V \cup A_1) \), we can extend \( F[1]|_V \) to a continuous invertible \( N \)-tuple \( H_1 \) on \( V \cup A_1 \). Let \( F[1] \) be given by

\[ F[1](x) = \begin{cases} F[x] & \text{if } x \in V \\ H_1(x) & \text{if } x \in A_1 \\ H_1(\tau(x)) & \text{if } x \in \tau(A_1). \end{cases} \]

Then \( F[1] \) is well defined and so \( F[1] \) is an invertible \( N \)-tuple in the space \( C(V \cup A_1 \cup \tau(A_1), \tau) \).

Next, since \( \dim (V \cup A_1 \cup \tau(A_1) \cup A_2) \leq N \), we may extend \( F[1] \) to an invertible continuous \( N \)-tuple \( H_2 \) on \( (V \cup A_1 \cup \tau(A_1)) \cup A_2 \). Here we assume without loss of generality that \( A_2 \setminus (A_1 \cup \tau(A_1)) \neq \emptyset \). Let \( F[2] \) be given by

\[ F[2](x) = \begin{cases} F[x] & \text{if } x \in V \cup A_1 \cup \tau(A_1) \\ H_2(x) & \text{if } x \in A_2 \\ H_2(\tau(x)) & \text{if } x \in \tau(A_2). \end{cases} \]

Then \( F[2] \) is well defined and so \( F[2] \) is an invertible \( N \)-tuple in the space \( C(V \cup A_1 \cup \tau(A_1) \cup A_2 \cup \tau(A_2), \tau) \).
After $p$ steps, where $p \leq n$, we obtain the desired extension
\[ F^{[2]} := F_p^{[1]} \in U_N(C(X, \tau)) \]
of $F^{[1]}|_{V}$ and hence of $f|_{Z(g)}$.

6. Recovering previous results

As a corollary to our main theorem above, we obtain the result in [6] giving
a characterization of the Bass and topological stable ranks for the algebra
\[ C(K)_{\text{sym}} = \{ f \in C(K) : f(z) = \overline{f(z)} \} \]
of continuous real-symmetric functions on real-symmetric compacta $K \subseteq \mathbb{C}$.
Here $\tau$ is given by $\tau(z) = \overline{z}$.

**Corollary 6.1. — [6, Theorem 3.4]**

Let $K \subseteq \mathbb{C}$ be compact and real-symmetric. Then
1. $\text{bsr} C(K)_{\text{sym}} = \text{tsr} C(K)_{\text{sym}} = 1$ if and only if the interior $K^0 = \emptyset$ and $K \cap \mathbb{R}$ is totally disconnected or empty;
2. $\text{bsr} C(K)_{\text{sym}} = \text{tsr} C(K)_{\text{sym}} = 2$ if and only the interior $K^0 \neq \emptyset$ or $K \cap \mathbb{R}$ contains an interval.

**Proof.** — We have to apply Theorem 5.9 and the facts from dimension theory that $\dim K = 2$ if and only if $K^0 \neq \emptyset$ and $\dim K = 0$ if and only if $K$ is totally disconnected. \(\square\)

As a further corollary we obtain the multi-variable version in [7]. Here a compact set $K \subseteq \mathbb{C}^n$ is called real-symmetric if $z := (z_1, \ldots, z_n) \in K$ implies $\overline{z} := (\overline{z}_1, \ldots, \overline{z}_n) \in K$. If $\tau(z) = \overline{z}$, then $\tau$ is an involution on $K$ and $C(K)_{\text{sym}} = \{ f \in C(K) : f(z) = \overline{f(z)} \}$ therefore coincides with $C(K, \tau)$. Note also that the set of fixed points of $\tau$ equals $\mathbb{R}^n \cap K$.

**Corollary 6.2. — [7, Theorem 5.6]**

Let $K$ be a real-symmetric, compact set in $\mathbb{C}^n$. Then
\[ \text{bsr} C(K)_{\text{sym}} = \text{tsr} C(K)_{\text{sym}} = \max \left\{ \left\lfloor \frac{\dim K}{2} \right\rfloor, \dim (K \cap \mathbb{R}^n) \right\} + 1. \]

Finally, we obtain as a corollary to Theorem 5.9 the following result from [5]. Recall here that $C(M(H^\infty))_{\text{sym}}$ is the algebra of all continuous functions on the spectrum $M(H^\infty)$ of $H^\infty$ that satisfy $f(z) = \overline{f(\overline{z})}$ in $D = \{ z \in \mathbb{C} : |z| < 1 \}$.

**Corollary 6.3. — [5, Theorem 4.3]**

$\text{tsr} C(M(H^\infty))_{\text{sym}} = \text{bsr} C(M(H^\infty))_{\text{sym}} = 2$. 
Proof. — Let \( \phi_a \) be the evaluation functional \( f \mapsto f(a) \), where \( a \in \mathbb{D}, f \in H^\infty \). As was shown in [5, Corollary 1.3],
\[
C(M(H^\infty))_{\text{sym}} = C(M(H^\infty), \tau),
\]
where \( \tau \) is the unique extension of the involution \( \tau_0 : \varphi_a \mapsto \varphi_{\tau} \) on \( \mathbb{D} \) to a topological involution \( \tau \) between \( M(H^\infty) \) and itself. This map \( \tau \) is given by \( \tau(m) = m^* \), where for \( m \in M(H^\infty) \), \( m^* \) is defined as \( m^*(f) = m(f^*) \), \( f^*(z) = \overline{f(z)} \), \( f \in H^\infty \). Note that \( m^* \in M(H^\infty) \). Moreover, the set of fixed points \( E \) of \( \tau \) was shown to be the closure of the interval \([-1, 1]\) in \( M(H^\infty) \). Finally, using the main result in [5] that \( \dim E = 1 \) and Suárez’s result that \( \dim M(H^\infty) = 2 \), [13], we obtain the assertion of the corollary from Theorem 5.9.

Acknowledgements

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References

[5] R. Mortini. The covering dimension of a distinguished subset of the spectrum \( M(H^\infty) \) of \( H^\infty \) and the algebra of real-symmetric and continuous functions on \( M(H^\infty) \), preprint.