A CLASSROOM NOTE

HARMONIC CONJUGATES IN ARBITRARY PLANAR DOMAINS

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It is well known that every harmonic function, \( u \), in a simply connected planar domain \( \Omega \) admits a harmonic conjugate \( v \); that is a harmonic function \( v \) for which \( f := u + iv \) is holomorphic in \( \Omega \). The case of the punctured disk \( \{ z \in \mathbb{C} : 0 < |z| < 1 \} \) and the function \( u(z) = \log |z| \) show that the assertion above is no longer true in arbitrary domains. The aim of this elementary note is to present a necessary and sufficient condition for a harmonic function defined on an arbitrary planar domain \( \Omega \) in order to have a harmonic conjugate. The result, I did not find in any textbook, does not seem to be widely known. Its proof is well suited for presentation in any introductory course on complex function theory.

To begin with, let \( \Omega_m \) be a finitely connected domain in \( \mathbb{C} \) bounded by \( m \) pairwise disjoint smooth Jordan curves and let \( u \) and \( v \) be two functions continuously differentiable on a neighborhood of the closure \( \overline{\Omega}_m \) of \( \Omega_m \). Then, by taking a suitable orientation of the boundary curves of \( \Omega_m \), it follows from Green’s theorem ([Ga], p. 390) that

\[
\int_{\Omega_m} (u \Delta v - v \Delta u) dxdy = \int_{\partial \Omega_m} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) d\sigma,
\]

where \( d\sigma \) denotes the differential with respect to arclength and \( \frac{\partial}{\partial n} \) the derivative with respect to the outer normal to the curve. In particular, if \( u \) is harmonic, that is \( \Delta u = 0 \), in a neighborhood of \( \overline{\Omega}_m \), then

\[
\int_{\partial \Omega_m} \frac{\partial u}{\partial n} d\sigma = 0.
\]

(To see this, just take \( v \equiv 1 \) and note that harmonic functions are at least twice continuously differentiable.)

Moreover, if \( u \) is a \( C^2 \)-function in a domain

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$\Omega \subseteq \mathbb{C}$, then it easily follows from (1) that $u$ is harmonic in $\Omega$ if and only

$$\int_{\partial D} \frac{\partial u}{\partial n} d\sigma = 0$$

for every small disk $D$ contained in $\Omega$. On the other hand, for harmonic functions $u$ in $D$, it may be that $\int_{\Gamma} \frac{\partial u}{\partial n} d\sigma \neq 0$ for some closed piecewise smooth curves $\Gamma$ in $D$. The following theorem shows that this happens exactly when $u$ does not have a harmonic conjugate.

**Theorem 1.** Let $\Omega$ be an arbitrary domain in $\mathbb{C}$ and let $u$ be harmonic in $\Omega$. Then a necessary and sufficient condition on $u$ in order that $u$ has a harmonic conjugate in $\Omega$ is that for any closed, piecewise smooth curve $\Gamma \subset \Omega$ we have

$$\int_{\Gamma} \frac{\partial u}{\partial n} d\sigma = 0. \tag{2}$$

**Proof.** As usual, let $u_x$ and $u_y$ denote the partial derivatives of $u$ with respect to the variables $x$ and $y$. Let $t = t(x,y)$ denote the oriented, normalized, tangent vector to the curve $\Gamma$ at the point $(x,y)$ and $n = n(x,y)$ the normal vector to $t$, obtained by rotating $t$ by 90 degree clockwise. Let $g = u_x - iu_y$. Since $u$ is harmonic, $g$ satisfies the Cauchy-Riemann differential equations; hence $g$ is holomorphic in $\Omega$. Let $z = x + iy$ and $dz = dx + idy$. Then

$$\int_{\Gamma} gdz = \int_{\Gamma} (u_x dx + u_y dy) + i \int_{\Gamma} (u_x dy - u_y dx) = \int_{\Gamma} \frac{\partial u}{\partial t} d\sigma + i \int_{\Gamma} \frac{\partial u}{\partial n} d\sigma. \tag{3}$$

Now suppose that $u$ admits a harmonic conjugate $v$ on $\Omega$. Then $f := u + iv$ is holomorphic in $\Omega$ and, by using the Cauchy-Riemann differential equations, we see that $g = f'$. Thus $g$ has a primitive and so the integral of $g$ over any closed curve is zero; in particular, by (3), $\int_{\Gamma} \frac{\partial u}{\partial n} d\sigma = 0$.

To prove the converse, fix $z_0 \in \Omega$. For $z \in \Omega$, let $\Gamma_z$ denote a piecewise smooth curve in $\Omega$ joining $z_0$ with $z$. Define a function $v$ on $\Omega$ by

$$v(z) = \int_{\Gamma_z} \frac{\partial u}{\partial n} d\sigma$$

(see also [Ma], vol II, p. 146). By our hypothesis, $v$ is well defined. Moreover, $v(z) = \int_{\Gamma_z} (u_x dy - u_y dx)$. Clearly, $v$ is continuous on $\Omega$. Next we show that the partial derivatives of $v$ exist and that $v_x = -u_y$ and $v_y = u_x$. Hence $v$ is infinitely often differentiable and the pair $(u, v)$ satisfies the Cauchy-Riemann differential equations. Moreover, by Schwarz, it follows that $\Delta v = v_{xx} + v_{yy} = -u_{xx} + u_{xy} \equiv 0$. Hence $v$ is harmonic. Thus $v$ is the harmonic conjugate we were looking for.

Fix $z \in \Omega$ and let $h$ be a nonzero real number, $|h|$ so small that the segment $[z, z + h]$ joining $z$ with $z + h$ is contained in $\Omega$. Look upon $z$ as a point in $\mathbb{R}^2$, say $z = (\xi, \eta)$, and choose the parametrization $(x(s), y(s)) = (\xi + sh, \eta), \quad (0 \leq s \leq 1)$, of the segment $[z, z + h]$. Then, as $h \to 0$,

$$\frac{v(z + h) - v(z)}{h} = - \int_0^1 u_y(\xi + sh, \eta) ds \to -u_y(\xi, \eta).$$

Thus $v_x$ exists and $v_x = -u_y$. Similiarly for $v_y$. \qed
For additional reading in the case of a finitely connected planar domain $\Omega$, we refer the reader to a paper of S. Axler [A], where a nice proof of the "Logarithmic Conjugation Theorem" is given: let $K_j$, $(j = 1, \ldots, N)$ be the bounded connected components of $\mathbb{C} \setminus \Omega$ and let $u$ be harmonic in $\Omega$. Then for any choice of points $a_j \in K_j$ there exists real numbers $c_j$ and a function $f$ holomorphic in $\Omega$, such that

$$u(z) = \text{Re} f(z) + \sum_{j=1}^{N} c_j \log |z - a_j| \quad (z \in \Omega).$$

For an abstract approach in the context of harmonic forms on Riemannian manifolds, the interested reader could consult [GK].

Finally we note that relation (2) is known in applied mathematics as the no flux condition (see e.g. [Ga], p. 90), but is often related to the finite connectedness of the domain. We refer the reader to [GR] for typical applications of this condition in the study of conductivity problems or Navier-Stokes equations.

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References


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