IDEALS OF DENOMINATORS IN THE
DISK-ALGEBRA

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Abstract. We show that there do not exist finitely generated, non-principal ideals of denominators in the disk-algebra $A(D)$. Our proof involves a new factorization theorem for $A(D)$ that is based on Treil’s determination of the Bass stable rank for $H^\infty$.

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Notation, background

Let $H^\infty$ be the uniform algebra of bounded analytic functions on the open unit disk $\mathbb{D}$ and let $A(\mathbb{D})$ denote the disk-algebra; that is the subalgebra of all functions in $H^\infty$ that admit a continuous extension to the Euclidean closure $\overline{\mathbb{D}} = \{ z \in \mathbb{C} : |z| \leq 1 \}$ of $\mathbb{D}$.

Let $\gamma = n/d$ be a quotient of two functions $n$ and $d$ in $A = H^\infty$ or $A = A(\mathbb{D})$. It is well known that every ideal of denominators $D(\gamma) = \{ f \in A : f\gamma \in A \}$ in $A = H^\infty$ is a principal ideal, since $H^\infty$ is a pseudo-Bzout ring; the latter means that each pair of functions in $H^\infty$ has a greatest common divisor (see [11]). The situation in $A(\mathbb{D})$ is completely different, due to the fact that $A(\mathbb{D})$ does not enjoy the property of being a pseudo-Bézout ring. For example $1 - z$ and $(1 - z) \exp(-\frac{1+z}{1-z})$ do not have a greatest common divisor. Answering several questions of Frank Forelli [3, 4], the first author could prove in his Habilitationsschrift [8] that any closed ideal in $A(\mathbb{D})$ is an ideal of denominators; that an ideal of denominators is closed if and only if $\gamma \in L^\infty(\mathbb{T})$; that the complement inside $\overline{\mathbb{D}}$ of the zero set

$Z(D(\gamma)) = \bigcap_{f \in D(\gamma)} \{ z \in \overline{\mathbb{D}} : f(z) = 0 \}$

of $D(\gamma)$ is the set of points $a$ in $\overline{\mathbb{D}}$ for which there exists a neighborhood $U$ in $\overline{\mathbb{D}}$ such that $|\gamma|$ admits a continuous extension to $U$; and that every

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ideal of denominators in $A(\mathbb{D})$ contains a function $f$ whose zero set equals the zero set of the ideal (one then says that $\mathcal{D}(\gamma)$ has the Forelli-property.) The proof of this last result was based on the approximation theorem of Carleman (see [5, p. 135]).

In the present note we shall be concerned with the question whether finitely generated, but non-principal ideals in $A(\mathbb{D})$ can be represented as ideals of denominators. It turns out that this is not the case. Our proof uses as main ingredient a deep result of Treil [14] that tells us that $H^\infty$ has the Bass stable rank one. This is a generalization going far beyond the corona theorem and tells us that whenever $(f, g)$ is a corona pair in $H^\infty$, that is whenever $|f| + |g| \geq \delta > 0$ in $\mathbb{D}$, then there exists $h \in H^\infty$ such that $f + hg$ is invertible in $H^\infty$. We actually need an extension of this found by the second author of this paper to algebras of the form

$$H^\infty_E = \{f \in H^\infty : f \text{ extends continously to } \mathbb{T} \setminus E\},$$

where $E$ is a closed subset of the unit circle $\mathbb{T}$. That result on the Bass stable rank of $H^\infty_E$ will be used to prove a factorization property of functions in $A(\mathbb{D})$, which will be fundamental to achieve our main goal of characterizing the finitely generated ideals of denominators in $A(\mathbb{D})$.

From the applications point of view, there is also a control theoretic motivation for considering the question of finding out whether there are ideals of denominators which are finitely generated, but not principal. Indeed, [10, Theorem 1, p.30] implies that if a plant is internally stabilizable, then the corresponding ideal of denominators is generated by at most two elements, and moreover, if an ideal of denominators corresponding to a plant is principal, then the plant has a weak coprime factorization. In light of these two results, our result on the nonexistence of nonprincipal finitely generated ideal of denominators in the disk-algebra implies that every internally stabilizable plant over the disk-algebra has a weak coprime factorization. Finally, since the disk-algebra is pre-Bézout [12], it also follows that every plant having a weak coprime factorization, possesses a coprime factorization [10, Proposition, p. 54]. Consequently, every internally stabilizable plant over the disk-algebra has a coprime factorization.

1. Factorization in $A(\mathbb{D})$

Cohen’s factorization theorem for commutative, non-unital Banach algebras $X$ tells us that if $X$ has a bounded approximate identity, then every $f \in X$ factors as $f = gh$, where both factors are in $X$ (see e.g. [1, p.76]). For $A(\mathbb{D})$ this may be applied to every closed ideal of the
form $X = \mathcal{I}(E, A(\mathbb{D})) := \{ f \in A(\mathbb{D}) : f|_E \equiv 0 \}$, whenever $E$ is a closed subset of $\mathbb{T}$ of Lebesgue measure zero (note that $(e_n)$ with $e_n = 1 - p^n_E$ is such a bounded approximate identity, where $p_E$ is a peak function in $A(\mathbb{D})$ associated with $E$; see [7, p. 80] for a proof of the existence of $p_E$). In the present paragraph we address the following question: Let $f \in A(\mathbb{D})$. Suppose that $f$ vanishes on $E \subseteq \mathbb{T}$ and that $E$ can be written as $E = E_1 \cup E_2$, where the $E_j$ are closed, not necessarily disjoint.

(1) Do there exist factors $f_j$ of $f$ such that $f = f_1 f_2$ and such that $f_j$ vanishes only on $E_j$?

A weaker version reads as follows:

(2) Do there exist factors $f_j$ of $f$ such that $f = f_1 f_2$ and such that $f_1$ vanishes only on $E_1$ and $f_2$ has the same zero set as $f$?

We will first answer question (2) above. The proof works along the model of [8, Proposition 2.3]. It uses the following lemma that is based on the approximation theorem of Carleman (see [5]):

Lemma 1.1. [8, Lemma 1.1] Let $I$ be an open interval. Then for every continuous function $u$ and every positive, continuous error function $\varepsilon(x) > 0$ on $I$ there exists a $C^1$-function $v$ on $I$ such that $|u - v| < \varepsilon$ on $I$.

We shall also give an answer to a variant of question (1) whenever the sets $E_j$ are disjoint closed subsets in $\overline{\mathbb{D}}$. That result will be the main new ingredient to prove our result on the ideals of denominators.

In the sequel, let $Z(f)$ denote the zero set of a function.

Theorem 1.2. Let $E$ be closed subset of $\mathbb{T}$ and suppose that $f|_E \equiv 0$ for some $f \in A(\mathbb{D})$, $f \neq 0$. Then there exists a factor $g$ of $f$ that vanishes exactly on $E$. Moreover, $g$ can be taken so that the quotient $f/g$ vanishes everywhere where $f$ does.

Proof. We shall construct an outer function $g \in A(\mathbb{D})$ with $Z(g) = E$ such that $|f| \leq |g|^2$ on $\mathbb{T}$. Then, by the extremal properties for outer functions (see [6]), $|f| \leq |g|^2$ on $\overline{\mathbb{D}}$. Hence $|f/g| \leq |g|$ on $\overline{\mathbb{D}} \setminus E$. Clearly this quotient has a continuous extension (with value 0) at every point in $E$. Thus $f = gh$ for some $h \in A(\mathbb{D})$. To construct $g$, we write $\mathbb{T} \setminus E$ as a countable union of open arcs $I_n$. Note that $f$ vanishes at the two (or in case $E$ is a singleton, a single) boundary points of $I_n$. Let $p_E$ be a peak function associated with $E$. Consider on $\mathbb{T}$ the continuous function $q = |f| + |1 - p_E|$. Then $|q| > 0$ on $I_n$, $Z(q) = E$ and $q = 0$ on the boundary points of $I_n$. If the outer function associated with $q$ would be in $A(\mathbb{D})$, we were done. But we are not able to prove that.
So we need to proceed as in [8, p. 22]. Let \( I_n = [a_n, b_n] \). Using Lemma 1.1, there exists functions \( u_n \in C^1(I_n) \) so that

\[
|u_n - q| \leq \frac{1}{2}|q| \quad \text{on } I_n,
\]

and \( u_n(a_n) = u_n(b_n) = 0 \). In particular,

\[
(1.1) \quad \frac{1}{2}|q| \leq |u_n| \leq \frac{3}{2}|q| \quad \text{on } I_n.
\]

Let \( u : \mathbb{T} \mapsto \mathbb{R} \) be defined by \( u = u_n \) on \( I_n \), \( n = 1, 2, \ldots \), and \( u = 0 \) elsewhere on \( \mathbb{T} \). Then \( u \in C(\mathbb{T}) \), \( u \geq 0 \), and by the left inequality in (1.1), \( \log u \in L^1(\mathbb{T}) \). Since \( u \in C^1(\mathbb{T} \setminus Z(q)) \) and \( u|_{Z(q)} \equiv 0 \), the outer function

\[
g(z) = \sqrt{2} \exp \frac{1}{2} \left( \frac{1}{2\pi} \int_0^{2\pi} e^{it} + z \log |u(e^{it})| \, dt \right)
\]

belongs by [12, p.52] to \( A(\mathbb{D}) \). It is clear that on \( \mathbb{T} \) we have \(|g|^2 = 2u \geq |f| \geq |q| \) and that \( g \) vanishes only on \( E \). Moreover, \(|f|/|g| \leq |g| \) shows that \( f/g \in A(\mathbb{D}) \) and that \( Z(f/g) = Z(f) \).

The following \((H^\infty, A(\mathbb{D}))\)-multiplier type result will yield our final factorization result (Theorem 1.4), that will be central to our study of ideals of denominators.

**Theorem 1.3.** Let \( E \) be a closed subset of Lebesgue measure zero in \( \mathbb{T} \) and let \( f \in H^\infty \) be a function that has a continuous extension to \( \mathbb{T} \setminus E \); i.e \( f \in H^\infty_E \). Suppose that 0 does not belong to the cluster set of \( f \) at each point in \( E \). Then there exists a function \( h \in H^\infty_E \), invertible in \( H^\infty \), so that \( fh \in A(\mathbb{D}) \).

**Proof.** Consider a peak function \( p_E \in A(\mathbb{D}) \) associated with \( E \). By assumption, the ideal \( I \) generated by \( f \) and \( 1 - p_E \) in \( H^\infty_E \) is proper. (Here we have used the corona theorem for \( H^\infty_E \) [2].)

Since \( H^\infty_E \) has the stable rank one ([13, Theorem 5.2]), there exist \( h \) invertible in \( H^\infty_E \) and \( g \in H^\infty_E \) such that \( hf + g(1 - p_E) = 1 \). Since the only points of discontinuity of \( g \) are located on \( E \), we see that \( g(1 - p_E) \in A(\mathbb{D}) \). Thus \( hf \in A(\mathbb{D}) \). \( \square \)

**Theorem 1.4.** Let \( f \in A(\mathbb{D}) \). Suppose that \( Z(f) = E_1 \cup E_2 \), where the \( E_j \) are two disjoint closed sets in \( \overline{\mathbb{D}} \). Then there exist factors \( f_j \) of \( f \) in \( A(\mathbb{D}) \) such that \( f = f_1f_2 \) and \( Z(f_j) = E_j \).

**Proof.** By assumption, \( 2\varepsilon := \text{dist}(E_1, E_2) > 0 \). Choose around each point \( \alpha \in E_1 \cap \mathbb{T} \) a symmetric open arc \( A \subseteq \mathbb{T} \) with center \( \alpha \) and length \( \varepsilon \). Due to compactness, there are finitely many of these arcs whose union covers \( E_1 \cap \mathbb{T} \). Let \( V \) be the union of these arcs. By combining two adjacent arcs, we may assume that \( V \) writes as \( V = \bigcup_{n=1}^N [\alpha_j, \beta_j] \),
the closures of the arcs \( I_j := [\alpha_j, \beta_j] \) being pairwise disjoint. We also have that \( \overline{V} \cap E_2 = \emptyset \) as well as \( E_1 \cap \partial V = \emptyset \).

We first consider the outer factor \( F \) of \( f \). Note that
\[
F(z) = \exp \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |F(e^{it})| \frac{dt}{2\pi}.
\]
Consider the factorization \( F = F_1 F_2 \), where
\[
F_1(z) = \exp \int_{t: e^{it} \in V} \frac{e^{it} + z}{e^{it} - z} \log |F(e^{it})| \frac{dt}{2\pi},
\]
and
\[
F_2(z) = \exp \int_{t: e^{it} \in \mathbb{T} \setminus V} \frac{e^{it} + z}{e^{it} - z} \log |F(e^{it})| \frac{dt}{2\pi}.
\]
Then the \( F_j \) have continuous extensions to every point in \( \mathbb{T} \setminus \partial V \). Also \( Z(F_j) = \bar{E}_j \cap \mathbb{T} \). Applying Theorem 1.3, there exists an invertible function \( h \in H^\infty_{\partial V} \) so that \( G_1 := F_1 h \in A(D) \). Since \( f = (F_1 h)(\frac{1}{n} F_2) \), we obtain that outside the zeros of \( F_1 \), that is outside \( E_1 \), the function \( \frac{1}{n} F_2 \) is continuous. Note that \( h \) is continuous on \( E_1 \) as well as \( F_2 \) itself. Hence, \( G_2 := \frac{1}{n} F_2 \in A(D) \). Thus \( F = G_1 G_2 \) satisfies \( Z(G_j) = E_j \cap \mathbb{T} \).

Now suppose that \( f \) has an inner factor \( \Theta = BS_\mu \). Let
\[
\sigma(\Theta) = \{ \sigma \in \overline{D} : \liminf_{z \to \sigma} |\Theta(z)| = 0 \}
\]
be the support of \( \Theta \). Note that \( \sigma(\Theta) \subseteq Z(f) = E_1 \cup E_2 \). Now we split the support of \( \Theta \) into the corresponding parts \( \Xi_1 := \sigma(\Theta) \cap E_1 \) and \( \Xi_2 := \sigma(\Theta) \cap E_2 \) and write \( \Theta \) as \( \Theta_1 \Theta_2 \). Then \( f_j = \Theta_j G_j \) gives the desired factorization. \( \square \)

2. IDEALS OF DENOMINATORS

**Notation:** Let \( A \) be a commutative unital algebra. For \( f_j \in A \), let
\[
\mathcal{I}(f_1, f_2, \ldots, f_N) = \left\{ \sum_{j=1}^N g_j f_j : g_j \in A \right\}
\]
denote the ideal generated by the functions \( f_j \) \((j = 1, \ldots, N)\). We also denote the principal ideal \( \mathcal{I}(f) \) by \( fA \).

If \( \gamma = n/d \) is a quotient of two elements \( n, d \) in \( A \setminus \{0\} \), then
\[
\mathcal{D}(\gamma) = \{ f \in A : f \gamma \in A \}
\]
is the ideal of denominators generated by \( \gamma \). If \( \gamma \in A \), then it is easy to see that \( \mathcal{D}(\gamma) = A \).

Finally, if \( I \) is an ideal in \( A(D) \), then \( Z(I) = \bigcap_{f \in I} Z(f) \) denotes the zero set (or hull) of \( I \).
The following two Lemmas are well known (see [9]) and work for quite general function algebras. For the reader’s convenience we present simple proofs.

**Lemma 2.1.** Let $I$ be an ideal in $A(\mathbb{D})$ and let $M$ be a maximal ideal containing $I$. Suppose that $I = IM$. Then $I$ is not finitely generated.

**Proof.** Suppose that $I = (f_1m_1, \ldots, f_Nm_N)$ for some $f_j \in I$ and $m_j \in M$. Then
\[
|f_k| \leq C_k \sum_{j=1}^{N} |f_jm_j| \leq C_k \left( \sum_{j=1}^{N} |f_j|^2 \right)^{1/2} \left( \sum_{j=1}^{N} |m_j|^2 \right)^{1/2}.
\]
Thus, for $C = \sum_{k=1}^{N} C_k^2$,
\[
\sum_{k=1}^{N} |f_k|^2 \leq C \left( \sum_{j=1}^{N} |f_j|^2 \right) \left( \sum_{j=1}^{N} |m_j|^2 \right).
\]
Hence $\sum_{j=1}^{N} |m_j|^2 \geq 1/C$ on $\mathbb{D} \setminus Z(I)$. Since $Z(I)$ is nowhere dense, we get this estimate to hold true on $\mathbb{D}$. But this is a contradiction, since all the $m_j$ vanish at some point. $\square$

**Lemma 2.2.** Let $I$ be an ideal in $A(\mathbb{D})$. Suppose that $Z(I) \subseteq \mathbb{D}$. Then $I$ is generated by a finite Blaschke product.

**Proof.** Due to compactness of $Z(I)$, we know that $Z(I)$ is finite (or empty). Let $Z(I) = \{a_1, \ldots, a_N\}$ and let $m_n$ be the highest multiplicity of the zero $a_n$ at which all functions in $I$ vanish. We claim that $I$ is generated by the Blaschke product $B$ associated with these $(a_n, m_n)$. In fact, the inclusion $I \subseteq \mathfrak{I}(B)$ is trivial, since $B$ divides every function in $I$. By construction, $\bigcap_{f \in I} Z(f/B) = \emptyset$. Due to compactness, there are finitely many functions $f_j \in I$ so that $\bigcap_{j=1}^{N} Z(f_j/B) = \emptyset$. By the corona theorem for $A(\mathbb{D})$, we have that $1 \in \mathfrak{I}(f_1/B, \ldots, f_N/B)$. Thus $B \in I$. $\square$

The following works for every commutative unital ring.

**Lemma 2.3.** Let $n, d$ be two functions in $A$ such that $\mathfrak{I}(n, d) = A$. Then $\mathfrak{D}(n/d) = dA$.

**Proof.** Let $x, y \in A$ be such that $1 = xn + yd$. Then $f = x(fn) + (fy)d$. Now let $f \in \mathfrak{D}(n/d)$. Hence $fn = ad$ implies that $f = x(ad) + (fy)d \in dA$. The reverse inclusion is trivial, since $d \in \mathfrak{D}(n/d)$. $\square$

Lemma 2.3 applies in particular to $A = A(\mathbb{D})$ if we assume that $Z(n) \cap Z(d) = \emptyset$. 
Corollary 2.4. Suppose that the greatest common divisor of two elements \( n \) and \( d \) in \( A(\mathbb{D}) \) is a unit. Then \( \mathfrak{D}(n/d) \) is a principal ideal.

Proof. Since \( A(\mathbb{D}) \) is a Pre-Bézout ring (see [12]) we have that \( \mathfrak{I}(n, d) = A(\mathbb{D}) \). The rest follows from Lemma 2.3 above.

Proposition 2.5. Let \( B \) be a finite Blaschke product and let \( f \in A(\mathbb{D}), f \not\equiv 0 \). Then \( \mathfrak{D}(B/f) \) is a principal ideal generated by a specific factor of \( f \).

Proof. Let \( b \) be the Blaschke product formed with the common zeros of \( B \) and \( f \) (multiplicities included). Consider the function \( F = f/b \) and \( B^* = B/b \). We claim that \( \mathfrak{D}(B/f) = \mathfrak{I}(F) \). In fact, we obviously have that \( \mathfrak{D}(B/f) = \mathfrak{D}(B^*/F) \). But \( F \) does not vanish at the zeros of \( B^* \); so, by the corona theorem for \( A(\mathbb{D}) \), \( \mathfrak{I}(B^*, F) = A(\mathbb{D}) \). By Lemma 2.3, we get that \( \mathfrak{D}(B/f) = \mathfrak{D}(B^*/F) = \mathfrak{I}(F) \).

Observation 2.6. Let \( I \) be an ideal in \( A(\mathbb{D}) \). Suppose that \( f \in I \) and that \( f = gh \), where \( g, h \in A(\mathbb{D}) \) and \( Z(g) \cap Z(I) = \emptyset \). Then \( h \in I \).

This follows from the fact that the maximal ideal space is \( \mathbb{D} \): indeed, the assumption implies that the ideal generated by \( g \) and \( I \) is the whole algebra; hence \( 1 = ag + r \) where \( a \in A(\mathbb{D}) \) and \( r \in I \). Thus \( h = a(gh) + hr \in I \).

Proposition 2.7. Let \( I = \mathfrak{D}(n/d) \) and \( J = \mathfrak{D}(d/n) \) be ideals of denominators in \( A(\mathbb{D}) \). Suppose that \( Z(J) \subseteq \mathbb{D} \). Then \( J \) and \( I \) are principal ideals.

Proof. If \( Z(J) \subseteq \mathbb{D} \), then, by Lemma 2.2, \( J \) is a principal ideal generated by a finite Blaschke product \( B \). Hence, as we will show, \( I \) is a principal ideal, too. In fact, let \( \gamma = \frac{n}{d} \). Suppose that \( J = \mathfrak{D}(d/n) = BA(\mathbb{D}) \). Since \( n \in J \), we have that \( n = BN \) for some \( N \in A(\mathbb{D}) \). Since \( B \in J \), \( Bd = kn = kBN \); so \( d = kN \). Thus \( \gamma = (BN)/(kN) = B/k \). Note that \( k \) and \( B \) have no common zeros inside \( \mathbb{D} \), otherwise \( J = \mathfrak{D}(k/B) \) would contain a factor of \( B \). Thus \( \mathfrak{I}(B, k) = A(\mathbb{D}) \). Hence, by Lemma 2.3, \( I = kA(\mathbb{D}) \).

Applying Theorem 1.4, we obtain the following

Proposition 2.8. Let \( I = \mathfrak{D}(n/d) \) be an ideal of denominators in \( A(\mathbb{D}) \). Suppose that \( Z(I) \cap Z(n) = \emptyset \). Then \( I \) is a principal ideal.

Proof. Let \( I = \mathfrak{D}(n/d) \). Without loss of generality we may assume that \( n \) and \( d \) have no common zeros (otherwise we split of the joint Blaschke product and use the fact that \( A(\mathbb{D}) \) has the \( F \)-property; that is that \( uf \in A(\mathbb{D}) \) implies that \( f \in A(\mathbb{D}) \) for any inner function \( u \).
Note that by our assumption, $Z(I) \subseteq Z(d) \subseteq Z(n) \cup Z(I)$, and that this union is disjoint. By Theorem 1.4 we may factor $d$ as $d = d_1d_2$, where $Z(d_1) = Z(I)$ and $Z(d_2) \cap Z(I) = \emptyset$. We claim that $I = I_1 := \mathfrak{D}(n/d_1)$. In fact, let $f \in I_1$. Then $fn = gd_1$ for some $g \in A(\mathbb{D})$. Then $(d_2f)n = g(d_1d_2) = gd$, and hence $d_2f \in \mathfrak{D}(n/d) = I$. But $Z(d_2) \cap Z(I) = \emptyset$. Thus by the observation 2.6 above, we have that $f \in I$. So $\mathfrak{D}(n/d_1) \subseteq \mathfrak{D}(n/d)$.

To prove the reverse inclusion, let $f \in \mathfrak{D}(n/d)$. Then $fn = hd$ for some $h \in A(\mathbb{D})$. Hence $fn = (hd_2)d_1$. So $f \in \mathfrak{D}(n/d_1)$. We conclude that $\mathfrak{D}(n/d_1) = \mathfrak{D}(n/d)$. Since $Z(d_1) \cap Z(n) = \emptyset$, we obtain from Lemma 2.3 that $I_1 (= I)$ is a principal ideal. \hfill \Box

Recall that for $\alpha \in \overline{\mathbb{D}}$, $M(\alpha) = \{f \in A(\mathbb{D}) : f(\alpha) = 0\}$ is the maximal ideal associated with $\alpha$.

Using Theorem 1.4 and its companion Proposition 2.8, we are now ready to prove our main result on the structure of finitely generated ideals of denominators in $A(\mathbb{D})$. We note that the result would hold for the Wiener algebra $W^+$ of all absolutely convergent power series in $\mathbb{D}$ as well, if Theorem 1.4 and Proposition 2.8 could be proven for $W^+$.

**Theorem 2.9.** Let $\gamma = n/d$ be a quotient in $A(\mathbb{D})$. Then the ideal of denominators, $\mathfrak{D}(\gamma) = \{f \in A(\mathbb{D}) : f\gamma \in A(\mathbb{D})\}$, is either a principal ideal or not finitely generated.

**Proof.** Associate with $I := \mathfrak{D}(\gamma)$ the set $J = \{\gamma f : f \in \mathfrak{D}(\gamma)\}$. Then it is straightforward to check that $J$ is an ideal in $A$, too. In fact, $J = \mathfrak{D}(1/\gamma)$.

Suppose that $J$ is not proper; then $Z(J) := \bigcap_{f \in \mathfrak{D}(\gamma)} Z(\gamma f) = \emptyset$. By compactness, there exist finitely many $f_j \in \mathfrak{D}(\gamma)$ so that $\bigcap_{j=1}^n Z(\gamma f_j) = \emptyset$. Hence $1 = \sum_{j=1}^n g_j(\gamma f_j)$ for some $g_j \in A(\mathbb{D})$. Then $1/\gamma \in A(\mathbb{D})$; hence $\gamma = 1/a$ for some $a \in A(\mathbb{D})$. Then $\mathfrak{D}(\gamma) = aA(\mathbb{D})$, the principal ideal generated by $a$.

Now suppose that $Z(J) \neq \emptyset$.

**Case 1.** $Z(J) \cap Z(I) \neq \emptyset$. Let $\alpha \in Z(I) \cap Z(J)$. Consider any $f \in I$. Then $fn = gd$ for some $g \in J$.

If $\alpha \in \mathbb{D}$, then $f = (z - \alpha)F$ and $g = (z - \alpha)G$. Hence $Fg = Gd$ and so $F \in I$. Thus $I = I \cdot M(\alpha)$.

If $\alpha \in \mathbb{T}$, then we use the fact that the maximal ideal $M(\alpha)$ contains an approximate unit and hence by the Cohen-Varopoulos factorization theorem [15], for any $f, g \in M(\alpha)$, there is a joint factor $h \in M(\alpha)$ of $f$ and $g$, say $f = hF$ and $g = hG$ for $F, G \in A(\mathbb{D})$. Hence $Fg = Gd$ and again $F \in I$. Thus, also in this case, $I = I \cdot M(\alpha)$.  

By Lemma 2.1 above, $I$ cannot be finitely generated.

Case 2. $Z(I) \cap Z(J) = \emptyset$. Then there exist $f, g \in I$ such that $1 = f + \frac{n}{d}g$. Hence $d = df + ng$ and so $d(1 - f) = ng$. Thus $\gamma = \frac{n}{d} = \frac{1-f}{g}$. Without loss of generality, we may assume that $I$ is proper. Let $\alpha \in Z(I)$. Since $g \in I$, we have that $Z(I) \subseteq Z(g)$. Hence $0 = g(\alpha)$ and (since $f \in I$), $f(\alpha) = 0$, too. So $Z(I) \cap Z(1 - f) = \emptyset$. By Proposition 2.8, $I$ is a principal ideal. $\square$

References

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