THE COVERING DIMENSION OF A DISTINGUISHED SUBSET OF THE SPECTRUM $M(H^\infty)$ OF $H^\infty$ AND THE ALGEBRA OF REAL SYMMETRIC AND CONTINUOUS FUNCTIONS ON $M(H^\infty)$

by

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Abstract. — We show that the covering dimension, $\dim E$, of the closure $E$ of the interval $[-1,1]$ in the spectrum of $H^\infty$ equals one. Using Suárez’s result that $\dim M(H^\infty) = 2$, we then compute the Bass and topological stable ranks of the algebra $C(M(H^\infty))_{\text{sym}}$ of real symmetric continuous functions on $M(H^\infty)$.

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Introduction

In recent years the real pendants to the classical complex function algebras $A(\mathbb{D}), A(K), H^\infty(\mathbb{D})$ have gained a certain interest due to their appearance in control theory. These are, for example, the algebras

$A(K)_{\text{sym}} = \{f \in C(K), f \text{ holomorphic in } K^\circ \text{ and } f(z) = \overline{f(\overline{z})} \text{ for all } z \in K\}$,

where $K$ is a real symmetric compact set in $\mathbb{C}$ (that is $K$ satisfies $z \in K \iff \overline{z} \in K$),

$A_R(\mathbb{D}) = \{f \in A(\mathbb{D}) : f \text{ real valued on } [-1,1]\}$,

and

$H^\infty_R = H^\infty_R(\mathbb{D}) = \{f \in H^\infty(\mathbb{D}) : f \text{ real valued on } ]-1,1[\}$

(see [16, 18, 25, 26, 31, 32]). If $D$ is the closed unit disk, then of course $A(D)_{\text{sym}} = A_R(D)$. The main feature in the papers referenced above was to give a determination of the Bass and topological stable ranks. In addition,
extension problems to invertible tuples of real symmetric functions in several complex variables were studied in \cite{17} for the real algebras

\[ C(K)_{\text{sym}} = \{ f \in C(K) : f(z_1, \ldots, z_n) = \overline{f(z_1, \ldots, z_n)} \}, \]

of complex valued continuous functions on real symmetric compact sets \( K \) in \( \mathbb{C}^n \).

In the present work we will determine the topological and Bass stable ranks of the algebra \( C(M(H^\infty))_{\text{sym}} \) of all continuous functions on the spectrum \( M(H^\infty) \) of \( H^\infty \) that satisfy \( f(z) = \overline{f(z)} \) in \( D = \{ z \in \mathbb{C} : |z| < 1 \} \). Note that in view of the corona theorem, \( D \) can be viewed of as a dense subset of \( M(H^\infty) \). We will call \( C(M(H^\infty))_{\text{sym}} \) the real symmetric algebra associated with \( C(M(H^\infty)) \). Let us point out that the trace of \( C(M(H^\infty))_{\text{sym}} \) in \( D \) is a proper subalgebra of the algebra \( C_b(D, \mathbb{C}) \) of all bounded, continuous and complex valued functions on \( D \).

From a topological view point, the space \( M(H^\infty) \) is a very bizarre space; it is a non-metrizable, connected, compact Hausdorff space of cardinal at least \( 2^c \) that is neither locally connected, nor path-connected \cite{1}. In particular, \( M(H^\infty) \) is not contractible. Its covering dimension, though, is small: it is two \((\cite{27})\).

A quite difficult problem is a concrete characterization of those continuous functions on \( D \) that admit a continuous extension to \( M(H^\infty) \). K. Hoffman showed in his fundamental work \cite{9} that \( C(M(H^\infty)) \) is the smallest uniformly closed subalgebra of \( C_b(\mathbb{D}, \mathbb{C}) \) that contains the (complex)-valued bounded harmonic functions. C. Bishop \cite{2} showed that \( f \in C_b(\mathbb{D}, \mathbb{C}) \) has a continuous extension to \( M(H^\infty) \) if and only if \( f \) is uniformly continuous with respect to the hyperbolic metric in \( D \) and for every \( \epsilon > 0 \) there is a Carleson contour \( \Gamma \) in \( D \) so that \( f \) is within \( \epsilon \) of a constant on each connected component of \( D \setminus \Gamma \).

Henceforth, we give a thorough discussion of the algebra \( C(M(H^\infty))_{\text{sym}} \). It can be looked upon as a non trivial standard model for the classical real function algebras \( C(X, \tau) \) presented for example in the monograph \cite{12} by Kulkarni and Limaye.

1. The algebra \( C(M(H^\infty))_{\text{sym}} \)

We first look at several properties of the underlying space \( M(H^\infty) \) that are relevant to the study of the algebra \( C(M(H^\infty))_{\text{sym}} \). Let \( f \in C(M(H^\infty)) \) and define \( f^* \) by \( f^*(z) = \overline{f(z)} \). If \( f \in H^\infty \), then \( f^* \in H^\infty \) and the operation \( \sigma \) given by \( \sigma(f) = f^* \) is an algebra involution on \( H^\infty \). It is well known that \( \{ f \in H^\infty : f = f^* \} \) coincides with the algebra \( H^\infty_{\mathbb{R}} \) defined above. We shall now introduce the associated involution on \( C(M(H^\infty)) \).
Lemma 1.1. — For \( m \in M(H^\infty) \), let \( m^* \) be defined as \( m^*(f) = \overline{m(f^*)} \), \( f \in H^\infty \). Then \( m^* \in M(H^\infty) \). Moreover, if \( (\varphi_{z_\alpha}) \) is a net of point functionals in \( \mathbb{D} \) that converges to \( m \in M(H^\infty) \), then \( (\varphi_{z_\alpha}) \) is a net of point functionals in \( \mathbb{D} \) that converges to \( m^* \in M(H^\infty) \).

Proof. — It is obvious that \( m^* \) is additive. Moreover, \( m^* \) is homogeneous because
\[
m^*(\lambda f) = \overline{m((\lambda f)^*)} = \overline{\lambda m(f^*)} = \lambda m^*(f)
\]
whenever \( f \in H^\infty \). Thus \( m^* \in M(H^\infty) \). Now if \( \varphi_{z_\alpha} \to m \), then
\[
\varphi_{z_\alpha}(f) = f(z_\alpha) = \overline{f^*(z_\alpha)} = \varphi_{z_\alpha}(f^*) \to m(f^*).
\]

\( \square \)

Lemma 1.2. — Let \( \tau_0 : \mathbb{D} \to \mathbb{D} \) be the involution \( a \mapsto \overline{\alpha} \). Then \( \tau_0 \) admits a unique extension to a topological involution \( \tau \) between \( M(H^\infty) \) and itself.

Proof. — Let \( \varphi_\alpha : f \mapsto f(a) \) be the evaluation functional associated with \( a \in \mathbb{D} \). For \( m \in M(H^\infty) \), consider the functional \( m^* \) given above. Define \( \tau \) at \( m \) by \( \tau(m) = m^* \). Note that \( \tau(\varphi_\alpha) = \varphi_{\overline{\alpha}} \). Hence \( \tau : M(H^\infty) \to M(H^\infty) \) is an involution between \( M(H^\infty) \) and itself. It remains to show that \( \tau \) is continuous on \( M(H^\infty) \). So let \( m_\alpha \) be a net in \( M(H^\infty) \) converging to \( m \). Then for \( f \in H^\infty \)
\[
\tau(m_\alpha)(f) = m^*_\alpha(f) = m_\alpha(f^*) \to m(f^*) = m^*(f) = \tau(m)(f).
\]

Thus \( \tau \) is a topological involution extending \( \tau_0 \).

\( \square \)

For a topological involution \( \tau \) on a compact Hausdorff space \( X \) let
\[
C(X, \tau) := \{ f \in C(X, \mathbb{C}) : f(\tau(m)) = \overline{f(m)} \text{ for any } m \in X \}
\]
be the classical real function algebra as given for example in [12, p. 27]. Using Lemma 1.2 above, we can now represent \( C(M(H^\infty))_{\text{sym}} \) as an algebra of this type:

Corollary 1.3. — Let \( \tau \) be the involution from Lemma 1.2. Then
\[
C(M(H^\infty))_{\text{sym}} = C(M(H^\infty), \tau).
\]

Proof. — Recall that \( C(M(H^\infty))_{\text{sym}} \) was defined to be the set of all functions \( f \) in \( C(M(H^\infty)) \) such that \( f(z^*) = \overline{f(z)} \) for all \( z \in \mathbb{D} \). Let \( f \in C(M(H^\infty))_{\text{sym}} \) and \( m \in M(H^\infty) \). The continuity of \( \tau \) implies that \( f \circ \tau \in C(M(H^\infty)) \) whenever \( f \in C(M(H^\infty)) \). If \( m \) is not point evaluation at some point in \( \mathbb{D} \) then, by the corona theorem, we choose a net \( z_\alpha \) in \( \mathbb{D} \) such that \( \varphi_{z_\alpha} \to m \). Then, by Lemma 1.2
\[
f(\tau(m)) = \lim f(\tau(\varphi_{z_\alpha})) = \lim f(z_\alpha) = \lim \overline{f(z_\alpha)} = \overline{f(m)}.
\]
So \( C(M(H^\infty))_{\text{sym}} \subseteq C(M(H^\infty), \tau) \). The other inclusion is trivial noticing that \( \tau \) restricted to \( \mathbb{D} \) is \( \tau_0 \).

\( \square \)
Observation 1.4. — $f \in C_b(\mathbb{D}, \mathbb{C})$ has a continuous extension $F$ to $M(H^\infty)$ if and only if $f^*$ has.

Proof. — This follows from the representation $f^* = (F \circ \tau)|_\partial$ and the fact that $\tau$ is continuous on $M(H^\infty)$ (Lemma 1.2). Another way to see this is to use Hoffman’s theory that states that $C(M(H^\infty))$ is the uniformly closed subalgebra of $C_b(\mathbb{D}, \mathbb{C})$ generated by bounded holomorphic functions and their complex conjugates.

In conformity with our previous notation, we keep on writing $f^*$ for the function $f \circ \tau$, whenever $f \in C(M(H^\infty))$; that is,

$$f^*(m) = \overline{f(m^*)},$$

where $m \in M(H^\infty)$.

In view of Corollary 1.3 and [12, Theorem 1.3.20] we have the following result on the structure of the maximal ideals of $C(M(H^\infty))_{\text{sym}}$ and their associated multiplicative linear functionals (see also [13] for the case of the algebra $A_\mathbb{R}(\mathbb{D})$).

Theorem 1.5. — Let $F = \{ m \in M(H^\infty) : \tau(m) = m \}$ be the set of fixed points of $\tau$. Then the following assertions hold:

i) An ideal $I$ in $C(M(H^\infty))_{\text{sym}}$ is maximal if and only if

$$I = I_m := \{ f \in C(M(H^\infty))_{\text{sym}} : f(m) = 0 \}$$

for some $m \in M(H^\infty)$. Moreover, $I_m = I_m^*$ for any $m$.

ii) $I_m$ has co-dimension 1 (in the real vector space $C(M(H^\infty))_{\text{sym}}$) if and only if $m \in F$;

iii) $I_m$ has co-dimension 2 if and only if $m \in M(H^\infty) \setminus F$.

iv) The only multiplicative $\mathbb{R}$-linear functionals $\phi : C(M(H^\infty))_{\text{sym}} \to \mathbb{R}$ are given by $\phi(f) = f(m)$, where $m \in F$. Their kernels are those maximal ideals $I_m$ that have co-dimension 1 in the real vector space $C(M(H^\infty))_{\text{sym}}$.

v) The remaining multiplicative $\mathbb{R}$-linear functionals have target space $\mathbb{C}$, (regarded as an algebra over $\mathbb{R}$), and are given by $\phi(f) = f(m)$ or $\phi(f) = \overline{f(m)}$, where $m \in M(H^\infty) \setminus F$. Their kernels are the maximal ideals $I_m$ that have co-dimension 2 in the real vector space $C(M(H^\infty))_{\text{sym}}$.

vi) $C(M(H^\infty))$ is the complexification of $C(M(H^\infty))_{\text{sym}}$. Each $q \in C(M(H^\infty))$ uniquely writes as $q = f + ig$, where $f, g \in C(M(H^\infty))_{\text{sym}}$. Here $f = (q + q^*)/2$ and $g = (q - q^*)/(2i)$.

vii) $\sigma(f) = f^*$ is a topological involution on $C(M(H^\infty))$.

Proof. — For the proof, we just note that if $\tau(m) = m$, then the evaluation functional $\phi_m$ on $C(M(H^\infty))_{\text{sym}} = C(M(H^\infty), \tau)$ satisfies

$$\phi_m(f) = f(m) = f(\tau(m)) = \overline{f(m)}.$$
Hence \( \phi_m \) is real valued and so the kernel has co-dimension 1.

On the other hand, if \( \tau(m) \neq m \), then there exists (by [12, Lemma 1.3.7]) a function \( f \in C(M(H^{\infty}), \tau) \) with \( f(m) = i \) and \( f(\tau(m)) = -i \). Thus the evaluation functional \( \phi_m \) is a surjection onto the real algebra \( \mathbb{C} \); hence its kernel has codimension 2.

The results now follow from [12, Theorem 1.3.20].

That the maximal ideal spaces of \( C(M(H^{\infty}))_{sym} \) and \( C(M(H^{\infty})) \) can be identified also follows from the fact that if \( (f_1, \ldots, f_N) \in C(M(H^{\infty}))^N_{sym} \), then a solution to the Bezout equation \( \sum_{j=1}^{N} q_j f_j = 1 \) in \( C(M(H^{\infty})) \) yields the solution \( \sum_{j=1}^{N} \frac{q_j^* q_j}{2} f_j = 1 \) of the associated Bezout equation in \( C(M(H^{\infty}))_{sym} \). Similarly, since \( H^{\infty}_R \) is a real subalgebra of \( C(M(H^{\infty}))_{sym} \) its maximal space can be identified with that of \( H^{\infty} \) alike. Note, however, that if \( m \) is a character of \( H^{\infty}_R \), then the maximal ideals \( \text{Ker } m \) and \( \text{Ker } m^* \) coincide even in the case where \( m \neq m^* \).

In the next section we will determine the set \( F \) of fixed points of \( \tau \).

## 2. The closure of the open unit interval in \( M(H^{\infty}) \)

Let \( E \) be the closure of \([-1, 1\) in \( M(H^{\infty}) \), \( M^+ \) the closure of \( \mathbb{D}^+ := \{ z \in \mathbb{D} : \text{Im } z > 0 \} \) in \( M(H^{\infty}) \) and \( M^- \) the closure of \( \mathbb{D}^- := \{ z \in \mathbb{D} : \text{Im } z < 0 \} \) in \( M(H^{\infty}) \). Finally, \( \mathbb{T}^+ = \{ e^{i\theta} : 0 < \theta < \pi \} \) and \( \mathbb{T}^- = \{ e^{i\theta} : -\pi < \theta < 0 \} \).

The goal in this section is to prove that \( M^+ \cap M^- = E \). To this end, we need a couple of lemmas.

For \( f \in C(M(H^{\infty})) \), we denote by \( Z(f) = \{ m \in M(H^{\infty}) : f(m) = 0 \} \) the zero set of \( f \). The set of points in \( M(H^{\infty}) \) with non-trivial Gleason parts will be denoted as usual, by \( G \) (see [8, 9]).

**Observation 2.1.** — Let \( f \in C(M(H^{\infty})) \). Then \( f^*(x) = \overline{f(x)} \) whenever \( x \in E \).

**Proof.** — Let \( (r_n) \) be a net in \([-1, 1\) that converges to \( x \). Then

\[
f^*(x) = \lim f^*(r_n) = \lim f(r_n) = \overline{f(x)}.
\]

\( \Box \)

Our subsequent results will be based on the following assertion. Recall that for a real or complex function algebra \( A \) with character space \( M(A) \) a compact set \( C \subseteq M(A) \) is said to be \( A \)-convex if \( C \) coincides with its \( A \)-convex hull

\[
\hat{C} = \{ m \in M(A) : |\hat{f}(m)| \leq \max_{\hat{C}} |\hat{f}|, \forall f \in A \},
\]

where \( \hat{f} \) denotes the Gelfand transform of \( f \).
**Theorem 2.2.** — Let $S$ be a closed subset of the closure $E$ of $]-1,1[$ in $M(H^\infty)$. Then $S$ is $H^\infty$-convex as well as $H^\infty_{\mathbb{R}}$-convex.

**Proof.** — Let $x \notin S$. Since $H^\infty$ is separating (see [27, p. 242]), there exists $f \in H^\infty$ such that $f(x) = 0$ and $f \neq 0$ on $S$. We may assume that $\|f\|_\infty \leq 1$. Let $g = ff^*$. Then $g \in H^\infty_{\mathbb{R}}$ and so $g$ is real valued on $E$. By definition of $g$, we actually have that $1 \geq g \geq 0$ on $E$. Since $f \neq 0$ on $S$, we obtain from $f^* = \overline{f}$ on $E$, that $\sigma := \min_S g > 0$. Now let $h = 1 - g$. Then $h \in H^\infty \subseteq H^\infty_{\mathbb{R}}$. $h(x) = 1$ and so

$$\max_S |h| = \max_S h \leq 1 - \sigma < |h(x)|,$$

Thus $x$ does not belong to the $H^\infty_{\mathbb{R}}$-convex closure of $S$. Therefore $S$ is $H^\infty_{\mathbb{R}}$-convex as well as $H^\infty$-convex. \hfill $\square$

**Lemma 2.3.** — Let $b$ be an interpolating Blaschke product all of whose zeros $z_n$ in $\mathbb{D}$ satisfy $\text{Im} \ z_n < 0$. Suppose that $Z(b) \cap E = \emptyset$. Then $Z(b) \cap M^+ = \emptyset$ and $\overline{Z(b^*)} \cap M^- = \emptyset$.

**Proof.** — Let $x \in M(H^\infty)$ satisfy $b(x) = 0$. We may assume that $x \notin \mathbb{D}$. Then $x \in G$ and $x$ belongs to the closure of the $\{z_n : n \in \mathbb{N}\}$ (see [8, p. 379]). Assuming that $x \in M^+ = \text{cl}(\mathbb{D}^+)$, we get from Hoffman’s result [9, p. 103] that $\rho(Z(b) \cap \mathbb{D}, \mathbb{D}^+) = 0$. For $j \in \mathbb{N}$, let $u_j \in \mathbb{D}^+$ and $n(j)$ be chosen so that $\rho(z_{n(j)}, u_j) \leq 1/j$. Then every cluster point $m$ of $\{z_{n(j)} : j \in \mathbb{N}\}$ belongs to $Z(b)$.

Note that $\text{Im} z_{n(j)} < 0$ and $\text{Im} u_j > 0$. By passing to subnets, we may assume that $z_{n(j)}(a) \to m$. Now, if $\text{Im} a < 0$ and $\text{Im} \xi \geq 0$, then

$$\rho(a, \text{Re} \ a) \leq \rho(a, \overline{a}) \leq \rho(a, \xi) + \rho(\xi, \overline{a}) \leq 2\rho(a, \xi).$$

Now letting $a = z_{n(j)(a)}$ and $\xi = u_{j(a)}$, we obtain that

$$\rho(z_{n(j)(a)}, \text{Re} z_{n(j)(a)}) \to 0.$$

By taking a further subnet, if necessary, $\text{Re} z_{n(j)(a)}$ then converges to some $m'$. Note that this implies that $m' \in E$. Since $\rho$ is semi-continuous [9, p. 103], $\rho(m, m') = 0$ and so $m = m'$. Thus $m \in E \cap Z(b)$. Hence $Z(b) \cap E \neq \emptyset$; a contradiction to our hypothesis. Therefore $x \notin M^+$. Since $x$ was an arbitrary zero of $b$, we conclude that $Z(b) \cap M^+ = \emptyset$. Due to symmetry, we obviously have that $Z(b^*) \cap M^- = \emptyset$, too. \hfill $\square$

Let us note that the previous result also holds for arbitrary Blaschke products. That proof though needs our result Theorem 2.7 that we are going to prove below. For the sake of completeness we present that argument, too, although we will not use this fact in the present paper.
Proposition 2.4. — Let $B$ be a Blaschke product all of whose zeros $z_\alpha$ in $\mathbb{D}$ satisfy $\text{Im} z_\alpha < 0$. Suppose that $Z(B) \cap E = \emptyset$. Then $Z(B) \cap M^+ = \emptyset$ and $Z(B^*) \cap M^- = \emptyset$.

Proof. — The hypothesis $Z(B) \cap E = \emptyset$ and the fact that $M^+ \cap M^- = E$ (Theorem 2.7) imply that $M^+ \cap Z(B)$ and $M^- \cap Z(B)$ are disjoint, open-closed sets in $Z(B)$. Suppose that both sets are nonempty. Then, by [10, Theorem 2.1], $B = B^+ B^-$, where $Z(B^+) = M^+ \cap Z(B)$, $Z(B^-) = M^- \cap Z(B)$. But $B$, and hence $B^+$, has no zeros in $\mathbb{D}^+$. Thus the factor $B^+$ does not exist. This contradiction shows that $Z(B) \cap M^+ = \emptyset$.

Finally we remark, that if $B$ is a Blaschke product all of whose zeros $z_\alpha$ in $\mathbb{D}$ satisfy $\text{Im} z_\alpha < 0$, then $Z(B) \cap M^+$ can be big, though. Just take the zeros $z_\alpha$ in $\mathbb{D}^-$ with $\rho(z_\alpha, 1 - \frac{1}{n^2}) \leq 1/n$ and let $B$ be the associated Blaschke product. Then $B(r) \to 0$ as $r \to 1$ and so $B$ vanishes identically on every Gleason part $P(x)$ associated with a point $x \in E \setminus \mathbb{D}$. We claim that, $(P(x) \cap M^+) \setminus E \neq \emptyset$ and $(P(x) \cap M^-) \setminus E \neq \emptyset$. In fact, suppose that $r_\alpha \to x$, $r_\alpha \in [0, 1]$. Then $w_\alpha := \frac{\alpha z_\alpha + x}{1 - \bar{z}_\alpha z_\alpha} \to L_x(z)$, $w_\alpha \in \mathbb{D}^+$ for $z \in \mathbb{D}^+$, $\overline{w_\alpha} \to (L_x(z))^*$ and $\overline{w_\alpha} \to L_x(z)$. Since the Hoffman map $L_x$ is a bijection, $L_x(z) \neq L_x(z)$, and so $L_x(z) \neq (L_x(z))^*$. Thus, by Corollary 2.9, $L_x(z) \in (P(x) \cap M^+) \setminus E$ and $L_x(z) \in (P(x) \cap M^-) \setminus E$. In particular, $\emptyset \neq P(x) \cap M^+ \subseteq Z(B)$.

Such a phenomenon does not occur when $b$ is an interpolating Blaschke product, since $Z(b) \subseteq M^-$ whenever the zeros in $\mathbb{D}$ are in the lower half-disk. Thus the fact that $M^+ \cap M^- = E$ implies that no point in $M^+ \setminus E$ can be a zero of $b$.

We return to the subject of the paper. Versions of the following function theoretic Lemma are well known. What we need here, are uniform estimates outside some cones. For the readers convenience we present its proof.

Lemma 2.5. — Let $u$ be the harmonic function with boundary values $1$ on $\mathbb{T}^+$ and $0$ on $\mathbb{T}^-$ and let $C_\kappa$ be the cone

$$C_\kappa = \{ z = x + iy \in \mathbb{D} : |y| \leq \kappa(1 - x) \}.$$ 

Then there exists $\sigma > 0$ such that $1 > u(z) \geq 3/4$ on

$$h^+(C_\sigma) := \{ z = x + iy \in \mathbb{D}, 0 \leq \sigma(1 - x) \leq y \}$$

and $0 < u(z) \leq 1/4$ on

$$h^-(C_\sigma) := \{ z = x + iy \in \mathbb{D}, y < 0, 0 \leq \sigma(1 - x) \leq |y| \}.$$ 

Moreover, $u(r) = 1/2$ for every $r \in ] - 1, 1[$, $1/2 \leq u \leq 1$ on $\mathbb{D}^+$ and $0 \leq u \leq 1/2$ on $\mathbb{D}^-$. 


Proof. — Note that $u$ has the form
\[ u(re^{i\theta}) = \frac{1}{2\pi} \int_{0}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos(t - \theta)} dt. \]

Now we use the following inequality:
\[ 1 + r^2 - 2r \cos s = (1 - r)^2 + 4r \sin^2(s/2) \leq (1 - r)^2 + s^2. \]
Let $\theta \in [0, \pi/4]$. Then
\[
\begin{align*}
\frac{1 + r}{1 - r} \int_{0}^{\theta} \frac{1}{1 + \left(\frac{\theta - t}{1 - r}\right)^2} dt + \frac{1 + r}{1 - r} \int_{\theta}^{\pi/2} \frac{1}{1 + \left(\frac{t - \theta}{1 - r}\right)^2} dt = \\
- \frac{1 + r}{2\pi} \arctan \left( \frac{\theta - t}{1 - r} \right) \bigg|_{0}^{\theta} + \frac{1 + r}{2\pi} \arctan \left( \frac{t - \theta}{1 - r} \right) \bigg|_{\theta}^{\pi/2} = \\
\frac{1 + r}{2\pi} \left[ \arctan \left( \frac{\theta}{1 - r} \right) + \arctan \left( \frac{\pi/2 - \theta}{1 - r} \right) \right].
\end{align*}
\]

Now if $re^{i\theta}$ stays outside the cone $C := \{z = \tilde{r}e^{i\tilde{\theta}} \in \mathbb{D} : |\tilde{\theta}| < C(1 - \tilde{r})\}$, then $C(1 - r) \leq \theta \leq \pi/4$ and hence
\[
u(re^{i\theta}) \geq \frac{1 + r}{2\pi} \left[ \arctan C + \arctan \left( \frac{\pi/4}{1 - r} \right) \right] - \frac{1}{2} + \frac{\arctan C}{\pi}.
\]
Now $C \subseteq C_\kappa$ with $\kappa = C = \tan \psi$, where $\psi$ is the angle between the horizontal axis and the line $y = \kappa(1 - x)$ respectively the curve $r(\theta) = 1 - \frac{\theta}{2}$ at the point 1. Thus $u(re^{i\theta}) \geq 3/4$ whenever $C$ is large, $r = r(C)$ close to 1, and $re^{i\theta} \in \mathbb{h}^+(C_\sigma)$ for some $\sigma = \sigma(C)$.

A change of variable $-t \to s$ shows that $u(z) + u(\overline{z})$ is the integral over the Poisson kernel on the whole interval $[0, 2\pi]$. Hence $u(z) + u(\overline{z}) = 1$. Now if $z \in \mathbb{h}^+(C_\sigma)$, then $\overline{z} \in \mathbb{h}^-(C_\sigma)$ and so
\[ u(\overline{z}) = 1 - u(z) \leq 1 - 3/4 = 1/4. \]
Finally, if $z = x$ is real, then $1 = u(z) + u(\overline{z}) = 2u(x)$; and so $u(x) = 1/2$. Next let $z \in \mathbb{D}^-$; that is $z = re^{i\theta}$ with $-\pi < \theta < 0$. Then $t - \theta \geq t$ and so
\[ u(re^{i\theta}) \leq \frac{1}{2\pi} \int_{0}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos(t)} dt = u(r) = 1/2. \]
If $z \in \mathbb{D}^+$, then $u(z) = 1 - u(\overline{z}) \geq 1/2$. \qed
Corollary 2.6. — Let $u$ be the harmonic function above. Then

1. $1/2 \leq u \leq 1$ on $M^+$;
2. $0 \leq u \leq 1/2$ on $M^-$
3. $u = 1/2$ on $E$

Question: do we have that $u = 1/2$ exactly on $E$?

Recall that for $\lambda \in T$, the fiber $M_\lambda$ is given by

$$M_\lambda = \{ m \in M(H^\infty) : m(z) = \lambda \},$$

where $z$ denotes the identity function here.

Theorem 2.7. — $M^+ \cap M^- = E$.

Proof. — First we note that $E \subseteq M^+ \cap M^-$ by definition. Now let $y \in M^+ \cap M^-$. Of course, we may assume that $y \notin \mathbb{D}$, since the fact that

$$M^+ \cap M^- \cap \mathbb{D} = ] -1, 1[$$

is obvious. We claim that $y$ belongs to one of the two fibers $M_1$ or $M_{-1}$. In fact, suppose that $y \in M_\lambda$, where $\lambda \notin \{-1, 1\}$. We may assume that $\text{Im} \lambda > 0$. Let $p_\lambda(z) = (1 + \overline{\lambda}z)/2$ be a peak function in $A(\mathbb{D})$ associated with $\lambda$. Then $p_\lambda \equiv 1$ on $M_\lambda \subseteq M^+$, but $|p_\lambda| \leq 1 - \eta < 1$ outside small neighborhoods of 1 within $\mathbb{D}$. In particular $|p_\lambda| \leq 1 - \eta$ on $M^-$. Hence $y \notin M^+ \cap M^-; a$ contradiction.

Let $x \in M(H^\infty) \setminus E$. Since $M^+ \cup M^- = M(H^\infty)$, we may assume that $x \in M^+$. Also, by the paragraph above, we may assume that $x \in M_1$. We claim that $x$ does not belong to $M^-$. 

Case 1 $x \in G$.

Choose a closed neighborhood $U$ of $x$ in $M(H^\infty)$ so that $U \cap E = \emptyset$. There exists an interpolating Blaschke product $b$ with $b(x) = 0$ such that $Z(b) \cap \mathbb{D} \subseteq U \cap \mathbb{D}$. Thus, by [8, p.379], $Z(b) \subseteq U$. Hence $Z(b) \cap E = \emptyset$. Decompose $b$ in a product $b = b_1 b_2 b_3$ of three interpolating Blaschke products, where the zeros of $b_1$ are those with imaginary part strictly positive, the zeros of $b_2$ are those with imaginary part strictly negative and where the zeros of $b_3$ are real.

Now $b_3(x) \neq 0$, since otherwise $x$ would lie in the closure of the zeros of $b_3$. But those are contained in $E$; a contradiction to the assumption that $Z(b) \cap E = \emptyset$.

Noticing that $Z(b_2) \cap E = \emptyset$, we obtain from Lemma 2.3 that $Z(b_2) \cap M^+ = \emptyset$. Thus $b_1(x) = 0$. Again, since $Z(b_1) \cap E = \emptyset$ and the zeros of $b_1$ are contained in $M^+$, we obtain from Lemma 2.3, that $Z(b_1) \cap M^- = \emptyset$. Thus $x \notin M^-$. Hence, $M^+ \cap M^- \cap G \subseteq E$.

Case 2 $x$ is a trivial point with $x \in M^+ \cap M_1$. 

Since the \( M(H^\infty) \)-closure \( S \) of every cone
\[
C_\kappa = \{ z = x + iy \in \mathbb{D} : x \geq 1/2, \ |y| \leq \kappa(1 - x) \}
\]
is contained in \( G \) (see [9, p. 108]), \( x \) is not in \( S \). By Lemma 2.5, the harmonic function \( u \) given there is bigger than \( 3/4 \) on \( h^+(C_\kappa) \) and smaller than \( 1/4 \) on \( h^-(C_\kappa) \) if \( C_\kappa \) has a sufficiently large opening. Now \( h^+(C_\kappa) \subseteq M^+, \ h^-(C_\kappa) \subseteq M^- \), \( h^+(C_\kappa) \cap h^-(C_\kappa) = \emptyset \) and
\[
M_1 = (h^+(C_\kappa) \cap M_1) \cup (S \cap M_1) \cup (h^-(C_\kappa) \cap M_1).
\]
Since \( x \in M^+ \) and \( u \geq 1/2 \) on \( M^+ \), we deduce that \( x \in h^+(C_\kappa) \) and so \( u(x) \geq 3/4 \). Also, since \( u \leq 1/2 \) on \( M^- \), we conclude that \( x \notin M^- \).

To sum up, we have shown that \( (M^+ \setminus E) \cap M^- = \emptyset \). Therefore \( M^+ \cap M^- = E \).

\textbf{Corollary 2.8.} Let \( m \in M(H^\infty) \). The following assertions hold:
1. \( m \in M^+ \) if and only if \( m^* \in M^- \).
2. The set \( E \) is the set of fixed points of \( \tau \); that is \( m = m^* \) if and only if \( m \in E \).
3. \( m \in E \) if and only if \( \overline{m(f^*)} = m(f) \) for any \( f \in H^\infty \).

\textbf{Proof.} (1) Let \( m \in M^+ \). By Lemma 1.1, if \( z_\alpha \to m, \ \text{Im} \ z_\alpha > 0 \), then \( \overline{z_\alpha} \to m^* \). Thus \( m^* \in M^- \).

(2) Let \( m = m^* \). By (1), \( m \in M^+ \cap M^- \). Using Theorem 2.7, we conclude that \( m \in E \). To prove the converse, let \( m \in E \). Choose a net \( r_\alpha \in ]-1,1[ \) converging to \( m \). Then, by Lemma 1.1, \( r_\alpha = \tau_\alpha \) converges to \( m^* \). Thus \( m = m^* \).

(3) This is merely a reformulation of the assertion that \( m = m^* \). □

We add the following additional information on \( u \).

\textbf{Proposition 2.9.} The following assertions hold:
1. \( u \equiv 1 \) on the set of trivial points in \( M^+ \);
2. \( u \equiv 0 \) on the set of trivial points in \( M^- \).

\textbf{Proof.} Since a trivial point \( x \) in \( M_1 \cap M^+ \) lies outside the closure of any cone, we can replace the number \( 3/4 \) in Lemma 2.5 by any number \( \sigma < 1 \) close to one. Thus \( u(x) \geq \sigma \) and so, by \( u(x) = 1 \). The same holds for \( x \in M^- \). □

3. The covering dimensions of \( E \) and \( M^+ \)

First let us recall the definition of the notion of covering dimension (or Čech-Lebesgue dimension) as given in [6, p. 54] or [20, p. 111]. Let \( X \) be a normal topological space. Then \( X \) is said to have dimension \( n \), denoted by \( \dim X = n \), if \( n \) is the smallest integer such that every finite open cover of \( X \)
has a finite open refinement of order \( n \). Here, as usual, the order of a family \( \mathcal{A} \) of subsets of \( X \) is the largest integer \( n \) such that \( \mathcal{A} \) contains \( n + 1 \) sets with a non-empty intersection.

In order to determine the covering dimension of \( E \), we need the following result from [20, p. 119]. Recall that a closed set \( C \) separates two disjoint closed sets \( E \) and \( F \) in a normal space \( X \) if \( X \setminus C = G \cup H \), where \( G \) and \( H \) are two disjoint open sets with \( E \subseteq G \) and \( F \subseteq H \).

**Proposition 3.1.** — If \( X \) is a normal space, the following assertions are equivalent:

1. \( \dim X \leq n \);
2. For each family of \( n + 1 \) pairs of closed sets
   \[
   \{(E_1, F_1), \ldots, (E_{n+1}, F_{n+1})\}
   \]
   where \( E_i \cap F_i = \emptyset \), there exists a family \( \{C_1, \ldots, C_{n+1}\} \) of closed sets such that \( C_i \) separates \( E_i \) and \( F_i \) and \( \bigcap_{i=1}^{n+1} C_i = \emptyset \).

**Theorem 3.2.** — a) Let \( E \) be the closure of \( ]-1,1[ \) in \( M(H^\infty) \). Then the covering dimension of \( E \) is one.

b) The covering dimension of the closure, \( M^+ \), of \( \{ z \in \mathbb{D} : \text{Im} z > 0 \} \) in \( M(H^\infty) \) is two.

**Proof.** — a) For \( j = 1, 2 \), let \( (E_j, F_j) \) be two pairs of disjoint closed sets in \( E \).
By Theorem 2.2, the sets \( E_j \cup F_j \) are \( H^\infty \)-convex. So the maximal ideal space of the algebras \( A_j = \overline{H^\infty}_{E_j \cup F_j} \) equals \( X := E_j \cup F_j \). Since \( E_j \) and \( F_j \) are open-closed in \( X \), Shilov’s idempotent theorem (see for example [7, p. 88]), yields a function \( g_j \in A_j \) such that \( g_j \equiv 1 \) on \( F_j \) and \( g_j \equiv 0 \) on \( E_j \). Thus there exists \( f_j \in H^\infty \) such that \( f_j \sim 1 \) on \( F_j \) and \( f_j \sim 0 \) on \( E_j \). Let \( h_j = f_j f_j^* \). Then \( h_j \in H^\infty_\mathbb{R} \) and \( h_j \) is real valued on \( ]-1,1[ \), hence on \( E \). Moreover, since for \( x \in E \) one has \( f^*(x) = \overline{f(x)} \) (2.1), \( h_j \) is close to 1 on \( F_j \) and close to 0 on \( E_j \). Let \( k_j = 2h_j - 1 \). Then \( k_j \in H^\infty_\mathbb{R} \) is real valued on \( E \), too, and \( k_j \) is close to 1 on \( F_j \) and close to \(-1\) on \( E_j \). Consider the pair \( (k_1, k_2) \). Since \( H^\infty_\mathbb{R} \) has the topological stable rank 2 ([18]), there is an invertible pair \( (g_1, g_2) \) of functions in \( H^\infty_\mathbb{R} \) so that \( g_j \) and \( k_j \) stay very close to each other. In particular, the \( g_j \) are real valued on \( E \) and \( g_j \) remains close to \(-1\) on \( E_j \) and close to 1 on \( F_j \). But \( Z(g_1) \cap Z(g_2) = \emptyset \). Thus we may choose \( C_j = Z(g_j) \cap E \) to conclude that \( C_j \) separates \( E_j \) and \( F_j \), (just take \( G_j = \{ x \in E : g_j < 0 \} \) and \( H_j = \{ x \in E : g_j > 0 \} \).) Hence, by Proposition 3.1, the covering dimension of \( E \) is less than or equal to one. The dimension cannot be zero, though, since \( E \) is a continuum. Thus \( \dim E = 1 \).

b) The fact that the covering dimension of the closure, \( M^+ \), of \( \{ z \in \mathbb{D} : \text{Im} z > 0 \} \) in \( M(H^\infty) \) is two follows from Suárez’s result [27] that
dim $M(H^\infty) = 2$ and the sum-property for the dimension \[6, \text{p.42, Theorem 1.5.3}\] that tells us that if $X$ is the union of a finite (or countably infinite) number of closed sets $X_j$ with $\dim X_j \leq d$, then $\dim X \leq d$. Here we have $X = M^+ \cup M^-$ and, due to symmetry, $\dim M^+ = \dim M^-$. \hfill \Box

Instead of using in the above proof the full power of the fact that $\text{tsr } H_\mathbb{R} = 2$, we can also prove part a) of Theorem 3.2 by applying the following Lemma.

**Lemma 3.3.** — Let $(k_1, k_2)$ be a pair of functions in $H_\mathbb{R}^\infty$. Then, for every $\epsilon > 0$ there exists a pair $(b_1 K_1, b_2 K_2)$ of functions in $H_\mathbb{R}$ such that

1. the $b_j$ are interpolating Blaschke products having only real zeros;
2. $b_1$ and $b_2$ have no common zeros on $M(H^\infty)$;
3. $K_1$ and $K_2$ are zero free on $E$.
4. $\|b_j K_j - k_j\|_E < \epsilon$.

**Proof.** — Let $k_j = B_j F_j$ be the Riesz factorization of $k_j$. Here $B_j$ is a Blaschke product and $F_j$ is zero free on $\mathbb{D}$. Since $k_j \in H_\mathbb{R}^\infty$, the zeros of $B_j$ are symmetric to the real axis and so $B_j$, as well as $F_j$, belong to $H_\mathbb{R}^\infty$. We may assume that $F_j \geq 0$ on $]-1,1[$. Then for $\epsilon > 0$, the functions $F_j + \epsilon$ have no zeros on $E$. Let $B_j = v_j u_j$, where $v_j$ is the Blaschke product formed with the real zeros of $B_j$. By \[15\], the Frostman shifts $w_j := \frac{v_j - \epsilon}{\overline{\epsilon} v_j}$ are Carleson-Newman Blaschke products. Write $w_j$ as $w_j = d_j e_j$, where $d_j$ is the factor of $w_j$ formed with the real zeros. Note that $w_j, d_j, e_j \in H_\mathbb{R}^\infty$. Since $d_j$ is a Carleson-Newman Blaschke product with real zeros only, it can be uniformly approximated by interpolating Blaschke products with real zeros. Let $W_j = e_j u_j$. Due to the symmetry of the zeros, $W_j(a) = 0$ if and only if $W_j(\overline{a}) = 0$. Therefore, for $r \in ]-1,1[$,

$$W_j(r) = \prod_{a : \text{Im} a > 0} \frac{\overline{a}}{|a|} \frac{a-r}{|1-ar|} \cdot \frac{a}{|\overline{a}|} \frac{\overline{a}-r}{|1-ar|} = \prod_{a : \text{Im} a > 0} \frac{|a-r|^2}{|1-ar|^2}.$$  

Thus $W_j \geq 0$ on $]-1,1[$. Hence $W_j + \epsilon$ is zero free on $E$. Thus we are able to approximate each $k_j$ by functions of the form $\tilde{b}_j K_j$, where $\tilde{b}_j$ is an interpolating Blaschke product with real zeros only and where

$$K_j = (F_j + \epsilon)(W_j + \epsilon).$$

Let $b_1 = \tilde{b}_1$. By moving those zeros of $\tilde{b}_2$ that are hyperbolically close to those of $b_1$, we may approximate $\tilde{b}_2$ by an interpolating Blaschke product $b_2$ so that $\inf_{|z|}(|b_1| + |b_2|) \geq \delta > 0$; for example by replacing $\tilde{b}_2 = b_2^{(1)} b_2^{(2)}$ by the interpolating Blaschke product $b_2^{(1)} \frac{b_2^{(2)} - \epsilon}{1 - \epsilon b_2^{(2)}}$. The tuple $(b_1 K_1, b_2 K_2)$ is now the desired item. \hfill \Box
4. The Bass and topological stable ranks for $C(M(H^{\infty}))_{\text{sym}}$

In this section we determine some $K$-theoretic data for the algebra $C(M(H^{\infty}))_{\text{sym}}$. Our construction will use the following lemma.

**Lemma 4.1.** — Let $q \in C(M^+, \mathbb{C})$. Suppose that $q$ is real-valued on $]-1, 1[$. Then $q$ admits a unique extension to $C(M(H^{\infty}))_{\text{sym}}$.

**Proof.** — Let $f$ be defined as

$$
f(m) = \begin{cases} 
q(m) & \text{if } m \in M^+, \\
q(m^*) & \text{if } m \in M^-.
\end{cases}
$$

Since $M^+ \cap M^- = E$ (Theorem 2.7) and $m = m^*$ on $E$ (Corollary 2.9), the real valuedness of $q$ on $]-1, 1[$, hence on $E$, implies that $f$ is well defined. Also, the continuity of $q$ on $M^+$ implies the continuity of $m \mapsto q(m^*)$ whenever $m \in M^-$. In fact, let $m_\alpha$ be a net in $M^-$ converging to $m$. Then $m_\alpha^* \tau(m_\alpha) \to \tau(m)$ by Lemma 1.2. Hence, using Corollary 2.9(1),

$$
q(m_\alpha^*) \to q(m^*).
$$

Thus $f$ is continuous on $M(H^{\infty})$. Since for $a \in \mathbb{D}$, $(\varphi_a)^* = \varphi_{\overline{a}}$, we obtain that $f \in C(M(H^{\infty}))_{\text{sym}}$. $\square$

Let $A$ be a commutative unital (real or complex) Banach algebra with unit element denoted by 1. The set of invertible $n$-tuples in $A$ is the set

$$
U_n(A) = \{ (f_1, \ldots, f_n) \in A^n \mid \exists g = (g_1, \ldots, g_n) \in A^n : \sum_{j=1}^n f_j g_j = 1 \}.
$$

An element $(f_1, \ldots, f_n, g) \in U_{n+1}(A)$ is said to be reducible, if there exists $(x_1, \ldots, x_n) \in A^n$ so that

$$(f_1 + x_1 g, \ldots, f_n + x_n g) \in U_n(A).$$

The smallest integer $n$ for which every element in $U_{n+1}(A)$ is reducible is called the **Bass stable rank** of $A$ and is denoted by $\text{bsr}(A)$. If no such integer exists, then $\text{bsr}(A) = \infty$. Many papers have dealt with the determination of the Bass and/or topological stable rank for concrete function algebras (see for instance [3, 4, 5, 11, 18, 22, 23, 24, 25, 26, 28, 29]).

A related concept is that of the **topological stable rank**, $\text{tsr}(A)$, of $A$ (see [21]). This is the smallest integer $n$ such that $U_n(A)$ is dense in $A^n$. If no such $n$ exists, then $\text{tsr}(A) = \infty$. It is well known that $\text{bsr}(A) \leq \text{tsr}(A)$ (see [21, 19]).

It has been shown by Vasershtein [30] and Rieffel [21] that whenever $X$ is a compact Hausdorff space, then

$$
\text{tsr}(C(X, \mathbb{C})) = \text{bsr}(C(X, \mathbb{C})) = \left[\frac{\dim(X)}{2}\right] + 1
$$
\[ \text{tsr}(C(X, \mathbb{R})) = \text{bsr}(C(X, \mathbb{R})) = \dim(X) + 1. \]

The following result can now be deduced from Theorem 3.2 and Suárez’s result [27] that the covering dimension of \( M(H^\infty) \) is 2.

**Corollary 4.2.**

1. \( \text{tsr}(C(M(H^\infty))) = \text{bsr}(C(M(H^\infty))) = 2; \)
2. \( \text{tsr}(C(E, \mathbb{R})) = \text{bsr}(C(E, \mathbb{R})) = 2; \)
3. \( \text{tsr}(C(M^+, \mathbb{C})) = \text{bsr}(C(M^+, \mathbb{C})) = 2. \)

For an \( n \)-tuple \( f = (f_1, \ldots, f_n) \) of complex-valued functions, let

\[ |f| = \left( \sum_{j=1}^{n} |f_j|^2 \right)^{1/2}. \]

As usual, \( S^n \) denotes the unit sphere

\[ \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} x_j^2 = 1\} \]

in \( \mathbb{R}^{n+1} \) sic! Finally, if \((z_1, z_2) \in \mathbb{C}^2 \) with \(|z_1|^2 + |z_2|^2 = 1\), then we say that \((z_1, z_2) \in S^3\).

We are now able to prove the main result of this paper. For matter of comparison, recall that \( \text{bsr}(H^\infty) = 1 \) ([29]), \( \text{tsr}(H^\infty) = 2 \) ([28]) and \( \text{bsr}(H^\infty_{\mathbb{C}}) = \text{tsr}(H^\infty_{\mathbb{C}}) = 2 \) ([18]).

**Theorem 4.3.**

\( \text{tsr}(C(M(H^\infty)))_{\text{sym}} = \text{bsr}(C(M(H^\infty)))_{\text{sym}} = 2. \)

**Proof.** — This follows as in [16] by using that \( \text{bsr}(C(E, \mathbb{R})) = 2 \) and \( \text{bsr}(C(M^+, \mathbb{C})) = 2. \) For the reader’s convenience we present those parts that need replacing \( D \) by \( M(H^\infty) \), and \( D^+ \) by \( M^+ \).

1. We first note that \( \text{bsr}(C(M(H^\infty)))_{\text{sym}} > 1 \), since the invertible pair \((z, 1 - z^2)\) is not reducible.

2. Next we indicate how to prove that \( \text{tsr}(C(M(H^\infty)))_{\text{sym}} \leq 2 \). Let \( f = (f_1, f_2) \in (C(M(H^\infty)))_{\text{sym}}^2 \) and

\[ E_n = \{m \in M^+ : |f(m)| \geq 1/n\}. \]

**Step 1** Suppose that \( E_n \cap E \neq \emptyset \). We claim that there is an \( \mathbb{R}^2 \)-valued extension of the tuple \( f/|f| \in C(E_n \cap E, S^1) \) to \( \tilde{f}_n \in C(E, S^1) \).

To prove this we choose \( g_n \in C(E, \mathbb{R}) \) with \( g_n \equiv 0 \) on \( E_n \cap E \) and \( g_n \equiv 1 \) on \( Z(f_1) \cap Z(f_2) \cap E \) (Urysohn’s Lemma).

Then the triple \((f_1, f_2, g_n)\) is invertible in \( C(E, \mathbb{R}) \). Since by Corollary 4.2 \( \text{bsr}(C(E, \mathbb{R})) = 2 \), there exist \( h_{1,n}, h_{2,n} \in C(E, \mathbb{R}) \) such that

\[ (f_1 + h_{1,n}g_n, f_2 + h_{2,n}g_n) \]
is invertible in $C(E, \mathbb{R})$. Now the pair

$$\tilde{f}_n := (f_1 + h_{1,n}g_n, f_2 + h_{2,n}g_n)/(f_1 + h_{1,n}g_n, f_2 + h_{2,n}g_n)$$

is the desired extension. We point out that $\tilde{f}_n$ is $\mathbb{R}^2$-valued.

If $E_n \cap E = \emptyset$, then we let $\tilde{f}_n = (1, 0)$.

**Step 2** Next we claim that there exists a $C^2$-valued extension of $f/|f| \in C(E_n, S^0)$ to $\hat{f}_n \in C(M^+, S^0)$ that coincides on $E$ with $\tilde{f}_n$.

In fact, define $F_n = (F_{1,n}, F_{2,n})$ by

\begin{align}
(4.1) & \quad F_n(m) = f(m)/|f(m)| \text{ whenever } m \in E_n, \\
(4.2) & \quad F_n(m) = \tilde{f}_n(m) \text{ whenever } m \in E
\end{align}

and extended continuously to $M(H^\infty)$ by Tietze. Note that $F_n$ is well defined, due to Step 1. Now let $G_n \in C(M^+, \mathbb{R})$ be a real valued continuous function with $G_n \equiv 0$ on $E_n \cup E$ and $G_n \equiv 1$ on $Z(F_{1,n}) \cap Z(F_{2,n})$. Then the triple $(F_{1,n}, F_{2,n}, G_n)$ is invertible in $C(M^+, \mathbb{C})$. Since by Corollary 4.2 $bbr(C(M^+, \mathbb{C})) = 2$, there exist $H_{1,n}, H_{2,n} \in C(M^+, \mathbb{C})$ such that

$$(F_{1,n} + H_{1,n}G_n, F_{2,n} + H_{2,n}G_n)$$

is invertible in $C(M^+, \mathbb{C})$. Now the pair

$$\hat{f}_n = (F_{1,n} + H_{1,n}G_n, F_{2,n} + H_{2,n}G_n)/(F_{1,n} + H_{1,n}G_n, F_{2,n} + H_{2,n}G_n)$$

is the desired extension.

**Step 3** It is easy to check that $|f - (|f| + 1/n)\hat{f}_n| \leq 3/n$ on $M^+$.

**Step 4** In the steps above we have found a $C^2$-valued function

$$g_n := (|f| + 1/n)\hat{f}_n$$

with $|f - g_n| \leq 3/n$ on $M^+$. Note that $g_n$ is $\mathbb{R}^2$-valued on $E \supseteq -1, 1]$. Thus by Lemma 4.1 we can use reflection to define a $C^2$-valued function $\Phi_n$ on $M$ (whose components are in $C(M(H^\infty))_{\text{sym}}$) so that $|f - \Phi_n| \leq 3/n$ on $M(H^\infty)$ and such that $|\Phi_n| \geq \frac{1}{n} > 0$ on $M(H^\infty)$.

It remains an open problem which pairs $(f, g)$ of functions in $C(M(H^\infty))_{\text{sym}}$ are reducible. Recall that in $H^\infty_{\mathbb{R}}$ an invertible pair $(f, g)$ is reducible if and only if $f$ has constant sign on the set $Z(g) \cap E$ (see [32, 33] and [14]). The situation in $C(M(H^\infty))_{\text{sym}}$ is more difficult, since a) the behaviour of $f$ outside $E$ is not determined by that in $E$ (in contrast to the analytic case) and b) the Bass stable rank of $C(M(H^\infty))$ is two, and not one. So a characterization of the reducible elements in $C(M(H^\infty))_{\text{sym}}$ must also involve conditions outside $M(H^\infty) \setminus E$. A necessary condition for example is the following:
Suppose that \((f, g)\) is reducible, say \(u = f + hg \neq 0\) on \(M(H^\infty)\) and let \(C\) be a connected component of \(M(H^\infty) \setminus Z(g)\). Suppose that the closure of \(C\) is contained in \(D\). Then \(u\) is a zero free (continuous) extension of \(f|_{\partial C}\) to \(C\). Thus the Brouwer degree of \(f\) satisfies \(d(f, C, 0) = d(u, C, 0) = 0\).

Necessary and sufficient criteria for reducibility of individual pairs even in \(C(D)\) are not known to the author; see [24] for related material.

5. Conjecture

In view of the results in this paper and the ones in [17], we conjecture that the following is true:

**Conjecture.** — Let \(X\) be a compact Hausdorff space, and \(\tau\) a topological involution of \(X\). Denote the set of fixed points of \(\tau\) by \(E\). Then

\[
\text{bsr} \ C(X, \tau) = \text{tsr} \ C(X, \tau) = \max \left\{ \left\lfloor \frac{\dim X}{2} \right\rfloor, \dim E \right\} + 1.
\]

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