GENERATING SETS FOR IDEALS OF FINITE TYPE IN $H^\infty$.

by

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Abstract. — We study ideals of finite type in $H^\infty$, that is ideals of the form $J(f_1, \ldots, f_n) = \{ f \in H^\infty | \exists C > 0 : |f| \leq C \sum_{j=1}^{n} |f_j| \text{ in } \mathbb{D} \}$ and give some necessary, respectively sufficient conditions for these ideals to be finitely generated. We also discuss finitely generated ideals of finite order $N$ and show that they are always generated by $N+1$ Blaschke products.

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Introduction

The well known Corona Theorem of Carleson tells us that the ideal $I = I(f_1, \ldots, f_n) = \{ \sum_{j=1}^{n} g_j f_j : g_j \in H^\infty \}$ equals the whole algebra, $H^\infty$, of bounded analytic functions on the unit disk $\mathbb{D}$ if and only if $\sum_{j=1}^{n} |f_j|$ is bounded away from zero in $\mathbb{D}$. Although a trivial necessary condition for a function $f \in H^\infty$ to belong to the ideal $I(f_1, \ldots, f_n)$ is that $|f| \leq \kappa \sum_{j=1}^{n} |f_j|$ for some constant $\kappa > 0$, this is far from being sufficient, as was already shown by Rao. In fact if $B$ and $C$ are Blaschke products without common zeros for which $\inf_{z \in \mathbb{D}} (|B(z)| + |C(z)|) = 0$, then $|BC| \leq |B|^2 + |C|^2$, but $BC \notin I(B^2, C^2)$. This motivated the introduction of the following class of ideals, that we will call ideals of finite type:

$$J = J(f_1, \ldots, f_n) = \{ f \in H^\infty | \exists C > 0 : |f| \leq C \sum_{j=1}^{n} |f_j| \text{ in } \mathbb{D} \}.$$

We shall also say that $J$ is the $J$-form of $I = I(f_1, \ldots, f_n)$.

Thus the Corona Theorem tells us that $1 \in J \iff 1 \in I$. Wolff's proof of the Corona Theorem yields that $f^3 \in I(f_1, \ldots, f_n)$ whenever $f \in J(f_1, \ldots, f_n)$ (see [1]). Only recently, Treil [20] showed that 3 is the best (integer-valued) power (see also [21]).

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On the other hand, as was shown in [5], \( f \in J \) implies that \( f^2 \in I \) whenever the hull of the ideal \( I \) does not meet the set of trivial points in the spectrum of \( H^\infty \). Again, by Rao's example, 2 is the best power here. See also Pau [15, 16] for related material.

In this paper we are now interested in describing various classes of ideals of finite type that coincide with the associated finitely generated ideals. So our leading question will be under which conditions \( I = J \), or more generally, when \( J(f_1, \ldots, f_n) \) is finitely generated? See also the survey [2]. Whereas the analogue problem in the disk-algebra \( A(D) \) has been completely settled (see [12]), only a couple of results are known in the setting of the algebra \( H^\infty \).

The first is due to von Renteln (the \( I \)-version) [17] and Tolokonnikov (the \( J \)-version) [18]. For a short proof see [11].

**Theorem 0.1.** — Suppose that \( I \) is finitely generated and that \( I \) or \( J \) contains an interpolating Blaschke product. Then \( I = J \).

In [5, Theorem 1.10], the following generalized Corona Theorem was established.

**Theorem 0.2.** — Suppose that \( f_1, f_2 \in H^\infty \) have no common factors. Let \( I = I(f_1, f_2) \) and \( J = J(f_1, f_2) \). Then the following assertions are equivalent:

1. \( I = J \);
2. \( \text{ord}(I,m) = 1 \) for every \( m \in Z(I) \);
3. \( I \) contains an interpolating Blaschke product;
4. \( J \) contains an interpolating Blaschke product;
5. \( |f_1(z)|^2 + (1 - |z|^2)|f_1'(z)| + |f_2(z)|^2 + (1 - |z|^2)|f_2'(z)| \geq \delta > 0 \) for every \( z \in \mathbb{D} \).

The situation for more than two generators or for higher order ideals remains a mistery. Classes of higher order ideals for which \( I = J \) were given in [13]:

**Theorem 0.3.** — Let \( B \) and \( C \) be interpolating Blaschke products and \( N \in \mathbb{N}^* = \{1, 2, \ldots \} \). Then

\[
I(B^N, B^{N-1}C, B^{N-2}C^2, \ldots, BC^{N-1}, C^N) = J(B^N, B^{N-1}C, B^{N-2}C^2, \ldots, BC^{N-1}, C^N) = J(B^N, C^N).
\]

In the present paper we want to investigate in how far powers of interpolating Blaschke products can be replaced by finite products of interpolating Blaschke products, so called Carleson-Newman Blaschke products, in order to achieve that \( I = J \). We shall also show that this property is quite sensitive on the order of the ideal \( I \) and the minimal number of generators for it. To this end, we first derive in section 2 the result that any finitely generated order-\( N \) ideal is generated by \( N + 1 \) functions. In section 4 we show that for order two functions \( b \) and \( c \) the ideal \( J(b,c) \) is three-generated, that is \( J(b,c) = I(f, g, h) \). This latter ideal then coincides with its \( J \)-form \( J(f, g, h) \) (which equals of course \( J(b,c) \)).

In section 5 we derive a necessary, respectively a sufficient condition, for \( N + 1 \)-generated ideals to coincide with their \( J \)-form. In section 6 we study more closely ideals of the form \( I = I(f^N, f^{N-1}g, \ldots, fg^{N-1}, g^N) \) for arbitrary functions and show that if \( I = J \), then the zero set of \( I \) cannot meet any trivial point in the spectrum of \( H^\infty \), that is \( Z(I) \subseteq G \). This will be
a result that supports the conjecture that \( I = J \) only if \( Z(I) \subseteq G \). Motivated by the result of Tolokonnikov [18] that \( J(f_1, \ldots, f_n) = \bigcap (I + bH^\infty) \), where the intersection runs through all interpolating Blaschke products, we shall present in section 7 several results on finitely generated ideals that can be represented as finite intersections of the form \( I + bH^\infty \) above, where the Blaschke product \( b \) is not necessarily interpolating.

In the first section we shall now present the necessary background for readers not familiar with the structure of the maximal ideal space of \( H^\infty \). A source of fine knowledge for this theory is John Garnett’s book [1].

1. Preliminaries concerning Carleson-Newman Blaschke products

Recall that a Blaschke product

\[
B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - a_n z} \quad \text{with} \quad \sum_{n=1}^{\infty} (1 - |a_n|) < \infty,
\]

is said to be interpolating if its zeros \((a_n)\) form an interpolating sequence; that is if for every \((w_n) \in \ell^\infty\) there exists \( f \in H^\infty \) such that \( f(a_n) = w_n \) for all \( n \). Carleson characterized those sequences by the condition

\[
\delta(B) := \inf_k \prod_{n=1, n \neq k}^{\infty} \rho_D(a_n, a_k) > 0,
\]

where \( \rho_D(z, w) = \left| \frac{z-w}{1 - wz} \right| \) is the pseudohyperbolic distance between two points \( z, w \in \mathbb{D} \). We call \( \delta(B) \) the uniform separation constant associated with \( B \). We also know that \( \delta(B) = \inf_k (1 - |a_k|^2) |B'(a_k)| \).

Another, more geometric characterization of interpolating sequences, also given by Carleson, involved the so-called Carleson measures. Just recall here that a Carleson measure \( \mu \) is a positive Borel measure on \( \mathbb{D} \) for which the embedding operator \( H^2 \to L^2_{\mu}(\mathbb{D}) \) is bounded. Here \( H^2 \) is the Hardy space. That characterization tells us that \((a_n)\) is interpolating if and only \( \inf_{n \neq m} \rho(a_n, a_m) > 0 \) (that is if \((a_n)\) is \( \rho \)-separated) and the measure \( \mu = \sum_n (1 - |a_n|^2) \delta_{a_n} \) is Carleson, where \( \delta_a \) denotes the Dirac measure at the point \( a \in \mathbb{D} \).

**Definition 1.** — A Blaschke product \( B \) whose zero sequence induces a Carleson measure is called a Carleson-Newman Blaschke product.

It is well known (see [9, 8]), that \( B \) is a Carleson-Newman Blaschke product if and only if \( B \) is a finite product of interpolating Blaschke products.

Let \( M(H^\infty) \) denote the spectrum of \( H^\infty \); that is the space of all nonzero, multiplicative linear functionals on \( H^\infty \) endowed with the weak-* topology \( \sigma((H^\infty)^*, M(H^\infty), H^\infty) \). As usual, we identify \( f \in H^\infty \) with its Gelfand transform \( \hat{f} \) defined on \( M(H^\infty) \) by \( \hat{f}(m) = m(f) \).

For two points \( x, m \) in \( M(H^\infty) \), we define the pseudohyperbolic distance of \( x \) to \( m \) by

\[
\rho(x, m) = \sup \{|f(m)| : f \in H^\infty, ||f||_\infty \leq 1, f(x) = 0 \}.
\]
It is well known that \( \rho(x, m) = \sup\{\rho_D(f(x), f(m)) : ||f||_\infty < 1\} \) and that the relation defined on \( M(H^\infty) \) by

\[
x \sim m \iff \rho(x, m) < 1
\]
defines an equivalence relation on \( M(H^\infty) \). The equivalence class containing a point \( m \) is called the Gleason part of \( m \) and is denoted by \( P(m) \). If the part, \( P(m) \), consists of a single point, we call the part (or point) trivial. If the part consists of more than one point, the part (or point) is called nontrivial. The set of all nontrivial points is denoted by \( G \). Obviously \( D \) is a part and \( D \subseteq G \). Hoffman’s theory [9] shows that for every Gleason part \( P(m) \) there is a continuous map \( L_m \) of \( D \) onto \( P(m) \) with \( L_m(0) = m \) such that \( f \circ L_m \) is analytic on \( D \) for all \( f \in H^\infty \). When the Gleason part of \( m \) is trivial, \( L_m \) is just a constant map. When \( P(m) \) is nontrivial, the map \( L_m \) is a bijection. Using this analytic representation of Gleason parts, it makes sense to define the order of a zero of \( f \) on the spectrum:

**Definition 2.** — Let \( m \in M(H^\infty) \) and suppose that \( f(m) = 0 \). Then \( n \in \mathbb{N} \) is said to be the order of the zero \( m \) of \( f \), denoted by \( \text{ord}(f, m) = n \), if the analytic function \( f \circ L_m \) has a zero of order \( n \) at 0. If \( f \circ L_m \) is the zero function, that is if \( f \) vanishes identically on the part \( P(m) \), then we let \( \text{ord}(f, m) = \infty \). If \( f(m) \neq 0 \), we say \( \text{ord}(f, m) = 0 \).

Hoffman [7, p. 79 and 100] showed that for \( m \in M(H^\infty) \) and \( f \in H^\infty \) with \( f(m) = 0 \) we have

\[
\text{ord}(f, m) = \sup\{n \in \mathbb{N} : f = f_1 \cdots f_n, f_j \in H^\infty, f_j(m) = 0 \text{ for } j = 1, 2, \cdots, n\}
\]

If \( m \) is a trivial point, then \( \text{ord}(f, m) = \infty \) whenever \( f(m) = 0 \). It is well known that a Blaschke product \( B \) is interpolating if and only if \( \text{ord}(B, m) = 1 \) for every zero \( m \) of \( B \) in \( M(H^\infty) \).

In the following definition, finite Blaschke product’s with simple zeros are considered to be interpolating Blaschke products. This notion was introduced in [3].

**Definition 3.** — The Newman degree of a finite product of interpolating Blaschke products, denoted \( \deg_{CN}(B) \), is the natural number given by

\[
\max\{\text{ord}(B, m) : m \in M(H^\infty)\}.
\]

In case of finite Blaschke products this reduces to saying that the highest multiplicity \( \max_\xi\text{ord}(B, \xi) \) of a zero \( \xi \) of \( B \) in \( \mathbb{D} \) is the Newman degree of \( B \). Also, \( \deg_{CN}(B) = 1 \) if and only if \( B \) is an interpolating Blaschke product.

The following result of Tolokonnikov relates the Newman degree of a finite product of interpolating Blaschke products to a (global) factorization property.

**Theorem 1.1 (Tolokonnikov [19]).** — Let \( B \) be a finite product of interpolating Blaschke products. Then the following are equivalent:

1. \( \deg_{CN}(B) = N_0 \);
2. \( N_0 = \min\{n \in \mathbb{N} : B = b_1 \cdots b_n\} \) where the \( b_j \) run through the set of all interpolating Blaschke products.
In the sequel we shall say that a Carleson-Newman Blaschke product has order $N$ if its Newman degree is $N$.

The zero set of $f$ in $M(H^\infty)$ is defined by

$$Z(f) = \{m \in M(H^\infty) : f(m) = 0\},$$

whereas its zero set in $\mathbb{D}$ is given by $Z_\mathbb{D}(f) = \{z \in \mathbb{D} : f(z) = 0\}$. The hull, or zero set of an ideal $I$ is the set $Z(I) = \bigcap_{f \in I} Z(f)$. Restricted to $\mathbb{D}$, we get $Z_\mathbb{D}(I) := \bigcap_{f \in I} Z_\mathbb{D}(f)$.

Next we define the higher order zero sets (see [4]). Let $N = \{0, 1, 2, \cdots\}$. For $n \in \mathbb{N}$ we let

$$E_n(f) = \{m \in M(H^\infty) : \text{ord}(f, m) \geq n\}.$$

Thus $E_1(f)$ is the zero set of $f$. The set of zeros of $f$ of infinite order is denoted by $Z^\infty(f)$. We observe that $Z^\infty(f) \subseteq E_{n+1}(f) \subseteq E_n(f) \subseteq \cdots \subseteq E_1(f)$.

Given an ideal $I$ and $n \in \mathbb{N}$, one can also define the higher order zero sets or hulls of ideals by $E_n(I) = \bigcap_{f \in I} E_n(f)$; that is, we let

$$E_n(I) = \{x \in M(H^\infty) : \text{ord}(f, x) \geq n \text{ for every } f \in I\}.$$

Note that $E_1(I)$ is the hull of $I$. If $I$ is an ideal in $H^\infty$, we let

$$\text{ord}(I, m) = \min\{\text{ord}(f, m) : f \in I\}.$$

Finally, $I$ is called an ideal of order $N$ ($N = 1, 2, \cdots, \infty$), denoted by $\text{ord } I = N$, if $\sup\{\text{ord}(I, m) : m \in M(H^\infty)\} = N$.

Let us mention that if $I$ is the principal ideal generated by the Carleson-Newman Blaschke product $B$, then $\deg_{CN}(B) = \text{ord } I$. Also, $N = \text{ord } I < \infty$ if and only if $E_{N+1}(I) = \emptyset$ and $E_j(I) \neq \emptyset$ for $j = 1, \ldots, N$.

2. Number of generators

We begin with several useful observations.

**Observation 2.1.** — Let $I(f_1, \ldots, f_m) = I(g_1, \ldots, g_n)$. Then $J(f_1, \ldots, f_m) = J(g_1, \ldots, g_n)$.

**Proof.** — Let $|f| \leq \sum_{j=1}^{m} |f_j|$. Since $f_j \in I(g_1, \ldots, g_n)$, we have $|f_j| \leq C_j \sum_{k=1}^{n} |g_k|$. Hence $|f| \leq m \max\{C_1, \ldots, C_n\} \sum_{k=1}^{n} |g_k|$. Thus $J(f_1, \ldots, f_m) \subseteq J(g_1, \ldots, g_n)$. Similarly for “≥”.

The following observation yields a wide class of ideals for which $I = J$.

**Observation 2.2.** — Suppose that $J = J(f_1, \ldots, f_m)$ is finitely generated, say $J = I(g_1, \ldots, g_n)$. Then $J(g_1, \ldots, g_n) = I(g_1, \ldots, g_n)$.

**Proof.** — Let $f \in J(g_1, \ldots, g_n)$. Since $g_j \in J(f_1, \ldots, f_m)$, we have that $|f| \leq C \sum_{k=1}^{m} |f_k|$. By hypothesis, $f \in I(g_1, \ldots, g_n)$. Thus $J(g_1, \ldots, g_n) \subseteq I(g_1, \ldots, g_n)$. The other inclusion is always true.
Note that, in general, we do not have \( J(f_1, \ldots, f_n) = I(f_1, \ldots, f_n) \) even if \( J \) is finitely generated. As an example we may take \( J = J(B^2, C^2) \). Then, by [13], \( J = I(B^2, C^2, BC) \), but \( I(B^2, C^2) \) is strictly contained in \( J(B^2, C^2) \).

**Observation 2.3.** — \( J = J(f_1, \ldots, f_n) \) is an ideal of order \( N \) if and only if \( I = I(f_1, \ldots, f_n) \) is an ideal of order \( N \).

**Proof.** — This follows immediately from the facts that \( Z^\infty(J) = Z^\infty(I) = \cap_{j=1}^n Z^\infty(f_j) \) and \( E_k(I) = E_k(J) \) for each \( k \).

The following theorems on finite order ideals were shown in [11, Theorem 1.4] and [4, Theorem 3.4]

**Theorem 2.4** ([11]). — An ideal \( I \) in \( H^\infty \) is generated (algebraically) by interpolating Blaschke products if and only if \( \text{ord}(I, m) = 1 \) for every \( m \in Z(I) \).

**Theorem 2.5** ([4]). — Let \( I \) be an ideal in \( H^\infty \) such that

\[
N = \text{ord} I := \sup \{ \text{ord}(I, x) : x \in Z(I) \} < \infty.
\]

Suppose that \( U_j \) are open sets satisfying \( E_j(I) \subseteq U_j \) \((j = 1, \ldots, N)\). Then \( I \) is algebraically generated by Carleson-Newman Blaschke products \( B \) of order \( N \) such that

\[
E_j(B) \subseteq U_j \quad (j = 1, \ldots, N).
\]

Our first result here will give additional information in the case of finitely generated ideals. One new feature will be the following: If \( \text{ord} I = N < \infty \), then it suffices to take \( N + 1 \) generators. But we will have more. These generators can take a very special form:

**Theorem 2.6.** — Let \( I = I(f_1, \ldots, f_m) \) be a finitely generated ideal of finite order \( N \) in \( H^\infty \), \( N \in \mathbb{N} \). Then \( I \) is generated by \( N + 1 \) Carleson-Newman Blaschke products of order \( N \). Moreover, fixing any function \( f \) in \( I \) of the form \( f = \prod_{j=1}^N B_j \), where the \( B_j \) are interpolating Blaschke products, then there exists a set of interpolating Blaschke products \( C_{k,j} \) \((k = 1, \ldots, N, j = 1, \ldots, k)\) such that

\[
I = I \left( \prod_{j=1}^N B_j, \ C_{1,1}(\prod_{j=1}^{N-1} B_j), \ C_{2,1}C_{2,2}(\prod_{j=1}^{N-2} B_j), \ \cdots, \ (\prod_{j=1}^{N-1} C_{N-1,j})B_1, \ \prod_{j=1}^N C_{N,j} \right).
\]

**Proof.** — By Theorem 2.5 above, we know that \( I \) is generated by Carleson-Newman Blaschke products of order \( N \). In particular, there exists a function \( B \) in \( I \) that has the form \( B = \prod_{j=1}^N B_j \), where the \( B_j \) are interpolating Blaschke products. We now proceed using ideas of Tolokonnikov [18]. Define the sign of a complex number \( z \) as \( \text{sgn} z = 1 \) if \( z = 0 \) and \( \text{sgn} z = e^{i \text{arg} z} \) if \( z \neq 0 \).

**Step 1.** Since the zero set \( Z\mathbb{D}(B_1) \) is an interpolating sequence, there exists \( g \in H^\infty \) such that \( g = \sum_{j=1}^m |f_j| \) on \( Z\mathbb{D}(B_1) \). Moreover, there are functions \( g_j \in H^\infty \) such that \( g_j = \text{sgn} \overline{f_j} \) on \( Z\mathbb{D}(B_1) \) for \( j = 1, \ldots, m \). It follows that \( g - \sum_{j=1}^m g_j f_j \) vanishes on \( Z\mathbb{D}(B_1) \); hence there
exists \( k_0 \in H^\infty \) such that \( g = \sum_{j=1}^{m} g_j f_j + k_0 B_1 \). Let \( h_0 := \sum_{j=1}^{m} g_j f_j \). Note that \( h_0 \in I \).

We claim that
\[
(2.1) \quad I = h_0 H^\infty + B_1 H^\infty \cap I.
\]

To see this, we first note that the inclusion "\( \supseteq \)" is trivial. So, in order to prove "\( \subseteq \)" let \( f \in I \). Then \( |f| \leq \kappa \sum_{j=1}^{m} |f_j| \) for some constant \( \kappa > 0 \). In particular, \( |f| \leq \kappa |g| \) on \( Z_{\emptyset}(B_1) \). Hence, by Carleson’s interpolation theorem, there are functions \( k_1 \) and \( h_1 \in H^\infty \) so that \( f = k_1 g + h_1 B_1 \). Hence
\[
f = k_1(h_0 + k_0 B_1) + h_1 B_1 = k_1 h_0 + (k_1 k_0 + h_1) B_1.
\]

Since \( f \in I \), we deduce that \( (k_1 k_0 + h_1) B_1 \in I \). So \( f \in h_0 H^\infty + B_1 H^\infty \cap I \). This proves our claim.

Clearly \( B_1 H^\infty \cap I = B_1 I_1 \) for some ideal \( I_1 \). Since by the McVoy-Rubel theorem \([10]\) the intersection of two finitely generated ideals in \( H^\infty \) is finitely generated, \( I_1 \) is finitely generated. Moreover, \( \prod_{j=2}^{N} B_j \in I_1 \) since \( \prod_{j=1}^{N} B_j \in I \cap B_1 H^\infty \). Also, \( \text{ord} I_1 = N - 1 \).

Now we show that one can replace \( h_0 \) by a certain Carleson-Newman Blaschke product of order \( N \). In fact, by \([4, \text{p. 122}]\), for small \( \varepsilon > 0 \), the function \( B + \varepsilon h_0 \) writes as \( C_0 F \), where \( C_0 \) is a Carleson-Newman Blaschke product of order equal or less than \( N \) and where \( F \) is invertible in \( H^\infty \). Since \( C_0 \in I \) and \( \text{ord} I = N \), we obtain that \( \text{deg}_{C_0} C_0 = N \). Using this function \( C_0 \), and the fact that \( \prod_{j=2}^{N} B_j \in I_1 \), it is easy to see that
\[
(2.2) \quad h_0 H^\infty + B_1 I_1 = C_0 H^\infty + B_1 I_1.
\]

So we are at the same situation as in the first step; \( I_1 \) replaces \( I \) and \( \prod_{j=2}^{N} B_j \) replaces \( \prod_{j=1}^{N} B_j \), and the order of the ideal has been reduced by 1.

Via induction, and by noticing that at the \( N \)-th step the ideal \( B_N H^\infty \cap I_{N-1} \) equals \( B_N H^\infty \) (since \( B_N \in I_{N-1} \)), we get that
\[
I = I \left( C_0, B_1 C_1, B_1 B_2 C_2, \ldots , \left( \prod_{j=1}^{N-1} B_j \right) C_{N-1}, \prod_{j=1}^{N} B_j \right),
\]

where the \( C_j \) are Carleson-Newman Blaschke products of order \( N-j \), \( j = 0, 1, \ldots , N - 1 \).

We note that it was previously shown by Tolokonnikov \([18]\) that if \( I \) is a finitely generated ideal containing an interpolating Blaschke product, then \( I \) is generated by two interpolating Blaschke products. Hence our Theorem 2.6 is a generalization of Tolokonnikov’s result.

**Corollary 2.7.** — Let \( J = J(f_1, \ldots , f_m) \) be an ideal of finite type and order \( N \) in \( H^\infty \).

Then there exist interpolating Blaschke products \( B_j \) \( (j = 1, \ldots , N) \) and Carleson-Newman Blaschke products \( C_k \) of order \( k \) \( (k = 1, \ldots , N) \), such that
\[
J = J \left( \prod_{j=1}^{N} B_j, C_1 \prod_{j=2}^{N} B_j, C_2 \prod_{j=3}^{N} B_j, \ldots , C_{N-1} B_1, C_N \right).
\]
Proof. — Let $I = I(f_1, \ldots, f_m)$. Then $I$ is an order $N$ ideal. By Theorem 2.6 above, there exist interpolating Blaschke products $B_j$ ($j = 1, \ldots, N$) and Carleson-Newman Blaschke products $C_k$ of order $k$ ($k = 1, \ldots, N$), such that

$$I = I \left( \prod_{j=1}^{N} B_j, C_1 \prod_{j=2}^{N} B_j, C_2 \prod_{j=3}^{N} B_j, \ldots, C_{N-1} B_1, C_N \right).$$

By the observation 2.1 above,

$$J = J \left( \prod_{j=1}^{N} B_j, C_1 \prod_{j=2}^{N} B_j, C_2 \prod_{j=3}^{N} B_j, \ldots, C_{N-1} B_1, C_N \right).$$

Next we will show that, in general, one cannot further reduce the number of generators. We need the following Lemma, developed in common with P. Gorkin in 1993.

Lemma 2.8. — Suppose that $C$ and $D$ are Blaschke products without common zeros in $\mathbb{D}$ such that for some functions $x_j \in H^\infty$

$$\sum_{j=0}^{N} x_j C^j D^{-j} \equiv 0.$$

Then, for every $j \in \{0, 1, \ldots, N\}$, we have that $x_j = 0$ on $Z(C) \cap Z(D)$.

Proof. — By assumption, $x_N C^N = -D \left( \sum_{j=0}^{N-1} x_j C^j D^{-j-1} \right)$. Since $D$ and $C$ have no common zeros in $\mathbb{D}$, $D$ divides $x_N$ (so $x_N$ vanishes on $Z(D) \supseteq Z(D) \cap Z(C)$), and there exists $y_1 \in H^\infty$ so that

$$\sum_{j=0}^{N-1} x_j C^j D^{-j-1} = y_1 C^N.$$

In particular,

$$x_{N-1} C^{N-1} + D \left( \sum_{j=0}^{N-2} x_j C^j D^{-j-2} \right) = y_1 C^N.$$

Hence

$$(x_{N-1} - y_1 C) C^{N-1} = -D \left( \sum_{j=0}^{N-2} x_j C^j D^{-j-2} \right).$$

As above, we conclude that $D$ divides $x_{N-1} - y_1 C$ (hence $x_{N-1} = 0$ on $Z(C) \cap Z(D)$), and that there exists $y_2 \in H^\infty$ so that

$$\sum_{j=0}^{N-2} x_j C^j D^{-j-2} = y_2 C^{N-1}.$$
Now proceeding inductively, we arrive at the conclusion that $x_1 = 0$ on $Z(C) \cap Z(D)$ and that $x_0 = y_NC$ for some $y_n \in H^\infty$. Thus $x_0 = 0$ on $Z(C) \cap Z(D)$, too.

We remark that in the Lemma above the interesting case is the one where the ideal $I(C, D)$ generated by $C$ and $D$ is proper, so that $Z(C) \cap Z(D) \neq \emptyset$.

**Theorem 2.9.** — Suppose that $C$ and $D$ are Blaschke products without common zeros in $\mathbb{D}$ such that $I(C, D)$ is a proper ideal. Let $I$ be the ideal

$$I(C^N, C^{N-1}D, C^{N-2}D^2, \ldots, CD^{N-1}, D^N).$$

If $\{f_1, \ldots, f_m\}$ is another set of generators for $I$, then $m \geq N + 1$; that is $N + 1$ is the minimal number of generators for $I$.

**Proof.** — By definition of the generating sets, there exist for every $i \in \{1, \ldots, m\}$ and $p \in \{0, \ldots, N\}$ functions $x_{j,k} \in H^\infty$, $(k = 0, 1, \ldots, N)$, such that $f_j = \sum_{k=0}^N x_{j,k}C^{N-k}D^k$, and functions $y_{p,\ell} \in H^\infty$ $(\ell = 1, \ldots, m)$ so that $C^{N-p}D^p = \sum_{\ell=1}^m y_{p,\ell}f_\ell$.

Let $\mathcal{C}$ be the matrix $\mathcal{C} = \begin{pmatrix} C^N & C^{N-1}D & \cdots & \cdots & CD^{N-1} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ D^N \end{pmatrix}$, $\mathfrak{F}$ the matrix $\mathfrak{F} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}$ and consider the

$$(m, N + 1)$$-matrix $X = \begin{pmatrix} x_{1,0} & \cdots & x_{1,N} \\ \vdots & \ddots & \vdots \\ x_{m,0} & \cdots & x_{m,N} \end{pmatrix}$, respectively the $(N + 1, m)$-matrix $Y = \begin{pmatrix} y_{0,1} & \cdots & y_{0,m} \\ \vdots & \ddots & \vdots \\ y_{N,1} & \cdots & y_{N,m} \end{pmatrix}$.

Then $\mathcal{C} = Y \cdot \mathfrak{F}$ and $\mathfrak{F} = X \cdot \mathcal{C}$, from which we conclude that $\mathcal{C} = (Y \cdot X) \cdot \mathcal{C}$. If we write the $(N + 1, N + 1)$-matrix $Y \cdot X$ as $(a_{k,\ell})_{k,\ell=0,\ldots,N}$, then for every $j \in \{0, \ldots, N\}$ we have

$$C^{N-j}D^j = \sum_{\ell=0}^N a_{j,\ell}C^{N-\ell}D^\ell,$$

and so, by putting $b_{j,\ell} = a_{j,\ell}$ if $\ell \neq j$ and $b_{j,\ell} = a_{j,\ell} - 1$ if $j = \ell$ we get that on $M(H^\infty)$

$$\sum_{\ell=0}^N b_{j,\ell}C^{N-\ell}D^\ell \equiv 0.$$

By Lemma 2.8 $b_{j,\ell} = 0$ on $Z(C) \cap Z(D)$. In particular, $a_{j,j} = 1$ on $Z(C) \cap Z(D)$. Since $I(C, D)$ is a proper ideal, $Z(C) \cap Z(D) \neq \emptyset$ and we can choose a single point $\xi \in Z(C) \cap Z(D)$.

Now evaluating the matrices $X$ and $Y$ at $\xi$, we see that $Y(\xi) \cdot X(\xi)$ is the identity matrix on $\mathbb{C}^{N+1}$. In particular, the linear mapping $\Phi_X : \mathbb{C}^{N+1} \to \mathbb{C}^m$ associated with $X(\xi)$ is injective. Thus $m \geq N + 1$. 

\hfill $\square$
3. Mixing zeros of generators

The following procedure of mixing the zeros of factors of a generating set of an ideal is quite astonishing.

**Proposition 3.1.** — Suppose that for \( j = 1, 2 \) the functions \( b_j \) and \( b_j^* \) are interpolating Blaschke products related in the following way: \( b_j = b_{1,j}b_{j,2} \), and \( b_j^* = b_{1,j}b_{2,j} \). Then \( I(b_1, b_2) = I(b_1^*, b_2^*) \).

**Proof.** — Since the \( b_j \) are interpolating Blaschke products, we see that \( I(b_{1,j}, b_{j,2}) = H^\infty \), (see [7]). So \( 1 = xb_{2,1} + yb_{2,2} \). Multiplying by \( b_1 = b_{1,1}b_{1,2} \) gives:

\[
b_1 = xb_{1,2}(b_{1,1}b_{2,1}) + yb_{1,1}(b_{1,2}b_{2,2}) = \hat{x}b_1^* + \hat{y}b_2^*.
\]

Hence \( b_1 \in I(b_1^*, b_2^*) \). Similar, \( b_2 \in I(b_1^*, b_2^*) \). Therefore \( I(b_1, b_2) \subseteq I(b_1^*, b_2^*) \). The reverse inclusion is shown in the same way. \( \square \)

For three generators we have a similar situation:

**Proposition 3.2.** — Let \( b_j \) and \( b_j^* \) be interpolating Blaschke products \( (j = 1, 2, 3) \) related in the following way: \( b_j = b_{1,j}b_{j,2}b_{j,3} \) and \( b_j^* = b_{1,j}b_{2,j}b_{3,j} \). Then \( I(b_1, b_2, b_3) = I(b_1^*, b_2^*, b_3^*) \).

**Proof.** — Since the factors \( b_{1,j}b_{2,j}b_{3,j}, b_{1,j}b_{2,j}, b_{1,j}b_{3,j} \) have no zeros in common on the spectrum of \( H^\infty \), the function 1 can be represented in the following ways (where \( x_{j,k} \in H^\infty \)):

\[
(3.1) \quad 1 = x_{1,1}b_{1,1}b_{2,1} + x_{1,2}b_{2,1}b_{3,1} + x_{1,3}b_{1,1}b_{3,1}
\]

\[
(3.2) \quad = x_{2,1}b_{1,2}b_{2,2} + x_{2,2}b_{2,2}b_{3,2} + x_{2,3}b_{1,2}b_{3,2}
\]

\[
(3.3) \quad = x_{3,1}b_{1,3}b_{2,3} + x_{3,2}b_{2,3}b_{3,3} + x_{3,3}b_{1,3}b_{3,3}.
\]

Then it is straightforward to verify that \( I(b_1^*, b_2^*, b_3^*) \subseteq I(b_1, b_2, b_3) \): indeed just write \( b_j^* \) as \( b_j^* \cdot 1 \cdot 1 \cdot 1 \) and replace the function 1 by the three representations above. The reverse inclusion is done similarly. \( \square \)

We guess that similar results hold for any number of generators.

4. When is \( J(f_1, \ldots, f_m) \) finitely generated?

In [5] it is shown that for two generators \( f \) and \( g \) in \( H^\infty \) the proper ideal \( I = I(f, g) \) coincides with \( J(f, g) \) if and only if the order of \( I \) is 1. As observed above (2.2), if \( J = J(f_1, \ldots, f_m) \) is finitely generated, say \( J = I(g_1, \ldots, g_N) \), then \( I(g_1, \ldots, g_N) = J(g_1, \ldots, g_N) \). In [13], this happened for ideals of the form \( J(C^N, D^N) \), where \( C \) and \( D \) were interpolating Blaschke products. Indeed, it was shown there that \( J(C^N, D^N) = I(C^N, C^{N-1}D, C^{N-2}D^2, \ldots, CD^{N-1}, D^N) \). Our next result shows that for functions of order 2, the ideal \( J(f, g) \) is finitely generated.

We need the following well known lemmas (see [1]).
Lemma 4.1 ([14]). — A Blaschke product $B$ is interpolating if and only if there exists a constant $\kappa > 0$ such that for all $z \in \mathbb{D}$ one has $|B(z)| \geq \kappa \inf \{\rho(z, \lambda) : \lambda \in Z_{\mathbb{D}}(B)\}$

Lemma 4.2. — If $B$ is a Carleson-Neuman Blaschke product, then every $\rho$-separated subsequence of its zero-sequence in $\mathbb{D}$ is an interpolating sequence.

Theorem 4.3. — Let $b$ and $c$ be two Carleson-Neuman Blaschke products of order 2. Then the ideal $J(b, c)$ is three-generated. Additionally, if $b$ and $c$ have no common zeros in $\mathbb{D}$, then there exists interpolating Blaschke products $B, B^*, C$ and $C^*$ such that

1. $b = BB^*, c = CC^*$,
2. $J(b, c) = I(BB^*, CC^*, BC)$.

In particular we have $J(BB^*, CC^*, BC) = I(BB^*, CC^*, BC)$.

Proof. — We shall factorize $b$ and $c$ each into two interpolating Blaschke products $B, B^*$ and $C, C^*$ such that

\[ \frac{B}{B^*} \text{ is bounded on } Z_{\mathbb{D}}(C^*) \quad \text{and} \quad \frac{C}{C^*} \text{ is bounded on } Z_{\mathbb{D}}(B^*). \]

Suppose we have done this, then, by Carleman's interpolation theorem, there exist $f \in H^\infty$ such that $B/B^* = f$ on $Z_{\mathbb{D}}(C^*)$. Hence $B - fB^*$ vanishes on $Z_{\mathbb{D}}(C^*)$ and so there is $g \in H^\infty$ such that $B = fB^* + gC^*$. That is $B \in I(B^*, C^*)$. By the same argument, $C \in I(C^*, B^*)$.

We claim that $|BC| \leq |b| + |c|$. In fact, since $B \in I(B^*, C^*)$, we have that $|B| \leq |B^*| + |C^*|$. Thus

\[ |BC| \leq |BB^*| + |CC^*| \leq |BB^*| + |CC^*| \text{ at points where } |C| \leq |B|. \]

Similarly, since $C \in I(C^*, B^*)$, we get that $|C| \leq |B^*| + |C^*|$. Hence

\[ |BC| \leq |BB^*| + |BC^*| \leq |BB^*| + |CC^*| \text{ at points where } |B| \leq |C|. \]

Thus $|BC| \leq |b| + |c|$. Let $f \in J(b, c)$ satisfy $|f| \leq |b| + |c| = |BB^*| + |CC^*|$. At the zeros $z_n$ of the interpolating Blaschke product $B$ (which is a divisor of $b$), we have $|f(z_n)| \leq |c(z_n)|$. Hence, by the interpolation argument above, $f \in I(c, B)$; that is $f = xc + yB$ for some $x, y \in H^\infty$.

Evaluating $y$ at the zeros $(w_n)$ of the interpolating Blaschke product $C$ yields that

\[ |y(w_n)| = \frac{|f(w_n)|}{|B(w_n)|} \leq \frac{|BB^*(w_n)|}{|B(w_n)|} = |B^*(w_n)|. \]

Hence $y \in I(B^*, C)$, say $y = aB^* + \tilde{a}C$. This implies that

\[ f = xc + (aB^* + \tilde{a}C)B \in I(b, c, BC). \]

Hence $J(b, c) \subseteq I(b, c, BC)$. On the other hand, since $|BC| \leq |b| + |c|$, we see that $I(b, c, BC) \subseteq J(b, c)$. Hence $J(b, c) = I(b, c, BC)$.

It remains to verify (4.1). By our assumption there exist 4 interpolating Blaschke products $b_1, b_2, c_1, c_2$ so that $b = b_1b_2$ and $c = c_1c_2$. Let $f = bc$. By Theorem 8.1, let $\delta$ be so small so that $U := \bigcup_{\xi, f(\xi) = 0} D_\rho(\xi, \delta)$ is a disjoint union of $n$-chains, $n \in \{1, 2, 3, 4\}$ and such that
$|f| \geq \eta > 0$ on $\mathbb{D} \setminus U$. We classify these chains into 8 classes (note that we take into account multiple zeros)

1. Class 1 consists of those chains containing a single zero of $b$,
2. Class 2 consists of those chains containing exactly a zero of $b$ and a zero of $c$,
3. Class 3 consists of those chains containing two zeros of $b$,
4. Class 4 consists of those chains containing two zeros of $b$ and one zero of $c$,
5. Class 5 consists of those chains containing two zeros of $b$ and two zeros of $c$,
6. Class 6 consists of those chains containing a single zero of $b$ and two zeros of $c$,
7. Class 7 consists of those chains containing two zeros of $c$,
8. Class 8 consists of those chains containing a single zero of $c$.

Within the Class 5 we consider the pseudohyperbolic distances between these zeros. Let us denote the two zeros of $b$ in the class 5 chain $C_n$ by $z_n$ and $z^*_n$ and the zeros of $c$ by $w_n$ and $w^*_n$. The $*$-zeros being those that satisfy

$$\rho(z^*_n, w^*_n) = \max\{\rho(z_n, w_n), \rho(z^*_n, w_n), \rho(z_n, w^*_n), \rho(z^*_n, w^*_n)\}. \tag{4.2}$$

Now we put in case 5, $z_n$ to the zero set of a Blaschke product that we call $B$, $z^*_n$ to the the zero set of a Blaschke product that we call $B^*$. Similarly, $w_n$ will be associated with a Blaschke product called $C$ and $w^*_n$ with $C^*$.

In case 1, the zero of $b$, denoted by $z_n$, will be associated with $B$, in case 2, the zeros $z_n$ of $b$ and $w_n$ of $c$ will be associated with $B$ respectively $C$, in case 3, one of the zeros of $b$ will be associated with $B$, the other with $B^*$, in case 4, the zeros of $b$ will be put to $B$ and $B^*$ respectively and the zero of $c$ to $C$, in case 6 the zero of $b$ is put to $B$, the zeros of $c$ to $C$ and $C^*$ respectively, in case 7 the zeros of $c$ will be put to $C$ and $C^*$ respectively and finally in case 8, the single zero of $c$ will be put to $C$.

By Lemma 4.2, these new Blaschke products $B, C, B^*, C^*$ are interpolating Blaschke products such that $b = BB^*$ and $c = CC^*$.

Now let $\gamma^* \in \mathbb{D}$ be a zero of $C^*$. We have to estimate the quotient $B(\gamma^*)/B^*(\gamma^*)$. By Vasjunin’s Lemma 4.1

$$\Delta := \frac{|B(\gamma^*)|}{|B^*(\gamma^*)|} \leq \frac{\inf_n \rho(\gamma^*, z_n)}{\kappa \inf_k \rho(\gamma^*, z^*_k)}. \tag{4.2}$$

If $\gamma^*$ does not belong to a disk $D_\rho(z^*_k, \delta)$ (and this happens in all the cases except 5) we get that $\Delta \leq \frac{1}{\kappa \delta}$. In the remaining case 5 we obtain that $\Delta \leq 1/\kappa$ because of the choice (4.2).

In the same way one can estimate the quotient $C/\mathbb{D}$ on the zeros of $B^*$ in $\mathbb{D}$. \hfill \Box

From section 2 we obtain that any finitely generated ideal $I$ of order two has the form $I = I(B_1 B_2, B_1 C_2, C_1 \tilde{C}_2)$ for some interpolating Blaschke products $B_j, C_j$ and $\tilde{C}_2$. In Theorem 4.3 above, $J(f, g)$ has the form $I(B_1 B_2, B_1 C_2, C_1 C_2)$: so the only difference between these representations is that $\tilde{C}_2$ is replaced by $C_2$. Our next Proposition shows that ideals of the form $I = I(B_1 B_2, B_1 C_2, C_1 C_2)$ where all the functions are interpolating Blaschke products, always coincide with their associated $J$ form.
Proposition 4.4. — For \( j = 1, 2 \) let \( B_j, C_j \) be interpolating Blaschke products. Then
\[
I := I(B_1B_2, B_1C_2, C_1C_2) = J(B_1B_2, B_1C_2, C_1C_2).
\]

Proof. — Without loss of generality, we may assume that \( Z_D(I) = \emptyset \). Let \( f \in \mathcal{H}^\infty \) satisfy
\[
|f| \leq |B_1B_2| + |B_1C_2| + |C_1C_2| \tag{4.3}
\]
Then on \( Z_D(C_2) \) we have \( |f| \leq |B_1B_2| \). Thus, \( f = xc_2 + yB_1B_2 \) for some functions \( x, y \in \mathcal{H}^\infty \)
(here we have used that \( C_2 \) is an interpolating Blaschke product). Dividing by \( C_1C_2 \) gives
\[
\frac{f}{C_1C_2} = \frac{yB_1B_2}{C_2} + \frac{x}{C_1} \tag{4.4}
\]
But on \( Z_D(B_1) \) the quotient \( \frac{x}{C_1} \) is bounded (by (4.3)). Hence, by (4.4), \( \frac{f}{C_2} \) is bounded on \( Z_D(B_1) \). So \( x \in I(C_1, B_1) \) (note that \( B_1 \) is an interpolating Blaschke product.) Hence \( f = xc_2 + yB_1B_2 \in I(C_1C_2, B_1C_2, B_1B_2) \).

An analysis of the proof of Theorem 4.4 shows that we need not to assume that \( C_1 \) and \( B_2 \) are interpolating Blaschke product. Thus we get that \( I(Bf, gC, BC) = J(Bf, gC, BC) \) for any \( f, g \in \mathcal{H}^\infty \) whenever \( B \) and \( C \) are interpolating Blaschke products. Note that this ideal can be generated by the following Carleson-Newman Blaschke products of order less than or equal to 2
\[
B(C + \varepsilon f), C(B + \varepsilon g), \text{ and } BC,
\]
whenever \( \varepsilon > 0 \) is small (see [4, 122]). As a concrete example we mention the ideals
\( I(B^n, C^m, BC) \), \( n, m = 1, 2, \cdots \). So we have \( I(B^n, C^m, BC) = J(B^n, C^m, BC) \) whenever \( B \) and \( C \) are interpolating Blaschke products.

Question 1. — Suppose that \( I \) is an order two ideal such that \( I = J \). Does \( I \) necessarily
has the form \( I(B_1B_2, B_1C_2, C_1C_2) \) for interpolating Blaschke products \( B_j, C_j \)?

5. Necessary respectively sufficient conditions for \( I = J \)

Theorem 5.1. — Let \( f_j, g_j \in \mathcal{H}^\infty \). Suppose that for all \((j, k)\) the functions \( f_j \) and \( g_k \) have no common factors. For \( n \in \mathbb{N} \), consider the ideals
\[
I_n = I \left( \prod_{j=1}^{n} f_j, \left( \prod_{j=1}^{n-1} f_j \right) g_n, \left( \prod_{j=1}^{n-2} f_j \right) g_{n-1} g_n, \cdots, f_1 \prod_{j=2}^{n} g_j, \prod_{j=1}^{n} g_j \right),
\]
and let \( J_n \) be the associated \( J \)-forms.

Suppose that \( I_N = J_N \). Then \( I_k = J_k \) for \( k = N - 1, \cdots, 2, 1 \). Moreover, \( I(f_j, g_j) \) is
generated by two interpolating Blaschke products for every \( j \in \{1, \ldots, N\} \).
Proof. — Step 1 We observe that $I_N = g_N I_{N-1} + (\prod_{j=1}^N f_j) H^\infty$. Let $f \in J_{N-1}$; that is

$$|f| \leq \kappa \left( \prod_{j=1}^{N-1} |f_j| + (\prod_{j=2}^{N-1} |f_j|)|g_{N-1}| + \cdots + |f_1| \prod_{j=2}^{N-1} |g_j| + \prod_{j=1}^N |g_j| \right).$$

Then

$$|fg_N| \leq \kappa \left( \prod_{j=1}^N |f_j| + (\prod_{j=1}^{N-1} |f_j|)|g_N| + \cdots + |f_1| \prod_{j=2}^N |g_j| + \prod_{j=1}^N |g_j| \right) \leq$$

$$\kappa \left( \prod_{j=1}^N |f_j| + (\prod_{j=1}^{N-1} |f_j|)|g_N| + \cdots + |f_1| \prod_{j=2}^N |g_j| + \prod_{j=1}^N |g_j| \right).$$

Thus $fg_N \in J_N$. Since $I_N = J_N$, we obtain that $fg_N \in I_N$; that is

$$fg_N \in g_N I_{N-1} + (\prod_{j=1}^N f_j) H^\infty.$$}

Since $g_N$ and $\prod_{j=1}^N f_j$ have no common factor,

$$f \in I_{N-1} + (\prod_{j=1}^N f_j) H^\infty \subseteq I_{N-1} + (\prod_{j=1}^{N-1} f_j) H^\infty \subseteq I_{N-1}.$$

Now induction downwards yields that $I_1 = J_1$. But $I_1 = I(f_1, g_1)$. By Theorem 0.2, $I_1$ contains an interpolating Blaschke product. Using Theorem 2.6, we get that $I_1$ is generated by two interpolating Blaschke products.

Step 2 For $m = 1, \ldots, N - 1$, we let $I_m^*$ be the ideal

$$I_m^* = I \left( \prod_{j=m}^N f_j, (\prod_{j=m}^{N-1} f_j)g_N, (\prod_{j=m}^{N-2} f_j)g_{N-1}g_N, \ldots, f_m \prod_{j=m+1}^N g_j, \prod_{j=m}^N g_j \right),$$

and $I_N^* = I(f_N, g_N)$. Note that $I_m^*$ has $N - m + 2$ generators. Moreover, $I_N^* = I_N$.

Now $I_m^*$ has exactly the same structure as $I_N$. The common factor of all but the last generator is $f_m$. Moreover, as we are going to show in step 3, the associated $J$-ideal will coincide with $I_m^*$. Hence, by Step 1, we get that $I(f_m, g_m) = J(f_m, g_m)$ for all $m \in \{1, \ldots, N\}$. As above, we see that $I(f_m, g_m)$ is generated by two interpolating Blaschke products.

Step 3 Using induction, it will be sufficient to show that the $J$-ideal $J_m^*$ associated with $I_m^*$ coincides with $I_m^*$. To this end, we observe that $I_1^* = f_1 I_2^* + (\prod_{j=1}^N g_j) H^\infty$. Now the proof proceeds exactly as above. \(\square\)

In Proposition 4.4 we showed that whenever $f_j, g_j$ are interpolating Blaschke products, then automatically $I_2 = J_2$. Do we have that $I_N = J_N$ for all $N$?

Our next result shows that under some additional assumptions we get a positive answer:
Proposition 5.2. — For $N \geq 3$ let $I$ be the ideal

$$I = I \left( \prod_{j=1}^{N} B_j, \left( \prod_{j=1}^{N-1} B_j \right) C_N, \left( \prod_{j=1}^{N-2} B_j \right) C_{N-1} C_N, \ldots, B_1 \left( \prod_{j=2}^{N} C_j \right), \prod_{j=1}^{N} C_j \right),$$

and $J$ the associated $J$-ideal, where the $B_j$ and $C_k$ are interpolating Blaschke products without common zeros in $\mathbb{D}$. Suppose that

$$(5.1) \quad \frac{C_j}{B_j} \text{ is bounded on } Z(B_{k+1}) \text{ for } j = 1, 2, \ldots, k \text{ and } k = 1, 2, \ldots, N - 2.$$

Then $I = J$.

Proof. — Let $f \in J$. Then $\frac{f}{\prod_{j=1}^{N} B_j}$ is bounded on $Z(C_N)$. Hence, $f = x \prod_{j=1}^{N} B_j + y C_N$ for some functions $x, y \in H^\infty$. Also, since $f \in J$, the quotient $q_1 := \frac{f}{\prod_{j=1}^{N} C_j}$ is bounded on $Z(B_1)$. But on $Z(B_1)$ we have $q_1 = \frac{y}{\prod_{j=1}^{N} C_j}$. Thus $y \in I(B_1, \prod_{j=1}^{N-1} C_j)$. We deduce that $f \in I(\prod_{j=1}^{N} B_j, \prod_{j=1}^{N} C_j, B_1 C_N)$; in other words $f = \tilde{x} \prod_{j=1}^{N} B_j + \tilde{y} \prod_{j=1}^{N} C_j + t B_1 C_N$ for some functions $\tilde{x}, \tilde{y}, t \in H^\infty$. Next, since $f \in J$ and $\frac{C_1}{B_1}$ is bounded on $Z(B_2)$, we get that $q_2 := \frac{f}{B_1 \prod_{j=2}^{N} C_j}$ is bounded on $Z(B_2)$, too. But on $Z(B_2)$ we have that

$$q_2 = \frac{t}{\prod_{j=2}^{N-1} C_j} + \frac{C_1}{B_1}.$$  

So $\frac{t}{\prod_{j=2}^{N-1} C_j}$ is bounded on $Z(B_2)$ and therefore $t \in I(B_2, \prod_{j=2}^{N-1} C_j)$. Thus

$$f \in I \left( \prod_{j=1}^{N} B_j, \prod_{j=1}^{N} C_j, B_1 \prod_{j=2}^{N} C_j, B_1 B_2 C_N \right).$$

The general scheme (induction hypothesis) will be the following: suppose that for all $\ell \leq N - 1$ we have shown that

$$f \in I \left( \prod_{j=1}^{N} B_j, \prod_{j=1}^{N} C_j, B_1 \prod_{j=2}^{N} C_j, \ldots, (\prod_{j=\ell-1}^{N} B_j)(\prod_{j=\ell-1}^{N} C_j), (\prod_{j=1}^{\ell-1} B_j) C_N \right).$$

In particular

$$f = R + t(\prod_{j=1}^{\ell-1} B_j) C_N$$

for some $R, t \in H^\infty$.

Then we look at the quotient

$$q_\ell := \frac{f}{B_1 \cdots B_\ell C_1 \cdots C_N}.$$
use that \( f \in J \), apply (5.1), and see that \( q_\ell \) is bounded on \( Z(B_\ell) \). From this, the representation \( f = R + t\prod_{j=1}^{\ell-1} B_j C_N \) and (5.1) we deduce that \( \frac{1}{f-\ell \cdot C_{N-1}} \) is bounded on \( Z(B_\ell) \). Hence

\[ t \in I(B_\ell, C_\ell, \ldots, C_{N-1}). \]

Therefore

\[ f \in I\left(\prod_{j=1}^{N} B_j, \prod_{j=1}^{N} C_j, B_1(\prod_{j=2}^{N} C_j), \ldots, (\prod_{j=2}^{\ell-1} B_j)(\prod_{j=1}^{N} C_j), (\prod_{j=1}^{\ell} B_j) C_N\right). \]

The procedure stops at \( \ell = N - 1 \).

\[ \square \]

6. Ideals of the form \( I(f^N, f^{N-1}g, \ldots, fg^{N-1}, g^N) \).

It is known from [13] that for two interpolating Blaschke products \( B \) and \( C \) we have

\[ I = I(B^N, B^{N-1}C, \ldots, BC^{N-1}, C^N) = J(B^N, B^{N-1}C, \ldots, BC^{N-1}, C^N). \]

Our next result will give the converse.

**Proposition 6.1.** — Let \( f, g \in H^\infty \) have no common factor. Suppose that

\[ I_N := I(f^N, f^{N-1}g, \ldots, fg^{N-1}, g^N) = J(f^N, f^{N-1}g, \ldots, fg^{N-1}, g^N) =: J_N. \]

Then there exist two interpolating Blaschke products \( b \) and \( c \) such that

\[ I_N = I(b^N, b^{N-1}c, \ldots, bc^{N-1}, c^N). \]

In particular, either \( I_N = H^\infty \) or \( Z(I_N) \subseteq G, \) and \( \text{ord} I_N = N. \)

**Proof.** — By Proposition 5.1 we have that

\[ I(f^{N-1}, f^{N-2}g, \ldots, g^{N-1}) = J(f^{N-1}, f^{N-2}g, \ldots, g^{N-1}). \]

Thus \( I_{N-1} = J_{N-1}. \) Backwards induction yields that \( I_1 = J_1, \) that is \( I(f, g) = J(f, g). \) Theorem 0.2 now yields that \( I(f, g) \) is an order one ideal. Hence, by Theorem 2.6, \( I(f, g) \) is generated by two interpolating Blaschke products \( b \) and \( c. \) It easily follows that \( I_N = I(b^N, b^{N-1}c, \ldots, bc^{N-1}, c^N). \)

\[ \square \]

**Corollary 6.2.** — Let \( f, g \in H^\infty \) have no common factor. Suppose that \( Z^\infty(f) \cap Z^\infty(g) \neq \emptyset. \) Then \( I(f^N, f^{N-1}g, \ldots, fg^{N-1}, g^N) \neq J(f^N, f^{N-1}g, \ldots, fg^{N-1}, g^N) \)

If the generators are powers of interpolating Blaschke products we get the following.

**Proposition 6.3.** — Let \( B_j \) be interpolating Blaschke products. Suppose that

\[ I := I(B^N_1, \ldots, B^N_m) = J(B^N_1, \ldots, B^N_m). \]

Then \( m \geq N + 1 \) and \( I = I(b^N, b^{N-1}c, \ldots, bc^{N-1}, c^N) \) for two interpolating Blaschke products \( b \) and \( c. \)
Proof. — By Theorem 2.6 we get that \(I(B_1, \ldots, B_m) = I(b, c)\) for two interpolating Blaschke products \(b\) and \(c\). Then clearly \(B_j^N \in I(b^N, b^{N-1}c, \ldots, bc^{N-1}, c^N) =: P_N\). Hence \(I \subseteq P_N\).

Now let \(f \in P_N\). Then, by [13], \(f \in J(b^N, c^N)\). But \(c = \sum_{j=1}^m x_j B_j\). Thus

\[
|c|^N \leq \kappa \sum_{j=1}^m |B_j|^N.
\]

Similarly for \(b\). Thus \(f \in J(B_1^N, \ldots, B_m^N)\). Hence

\[
P_N \subseteq J(B_1^N, \ldots, B_m^N) = I(B_1^N, \ldots, B_m^N) = I \subseteq P_N.
\]

Therefore, \(I = I(b^N, b^{N-1}c, \ldots, bc^{N-1}, c^N)\). By Theorem 2.9, \(m \geq N + 1\). \(\square\)

Such a situation considered in Proposition 6.3 really occurs, as the following result shows.

Proposition 6.4. — Let \(C\) and \(D\) be two interpolating Blaschke products. Then \(J(C^N, D^N)\) is generated by \(C^N, D^N\) and the functions \((C + \varepsilon_j D)^N\), where \(\varepsilon_j = e^{(i2\pi j)/N}\) for \(j = 1, \ldots, N - 1\). In particular,

\[
I(C^N, (C + \varepsilon_1 D)^N, \ldots, (C + \varepsilon_{N-1} D)^N, D^N) = \text{span}(C^N, (C + \varepsilon_1 D)^N, \ldots, (C + \varepsilon_{N-1} D)^N, D^N).
\]

Proof. — Since \(J(C^N, D^N) = I(C^N, C^{N-1} D, \ldots, CD^{N-1}, D^N)\) (see [13]), it is clear that \((C + \varepsilon_j D)^N \in J(C^N, D^N)\). In order to show the reverse inclusion, we will exhibit \(\lambda_{j,k} \in \mathbb{C}\), \((j, k = 0, \ldots, N)\), such that for \(0 \leq j \leq N\)

\[
(6.1) \quad C^{N-j} D^j = \sum_{k=1}^{N-1} \lambda_{j,k} (C + \varepsilon_k D)^N + \lambda_{j,0} C^N + \lambda_{j,N} D^N.
\]

Equivalently

\[
(6.2) \quad C^{N-j} D^j = \sum_{\ell=0}^{N-1} \binom{N}{\ell} \left( \sum_{k=1}^{N-1} \lambda_{j,k} \varepsilon_k^\ell \right) C^{N-\ell} D^\ell + \sum_{k=0}^{N-1} \lambda_{j,k} C^N + \sum_{k=1}^{N} \lambda_{j,k} D^N
\]

\[
(6.3) \quad = \sum_{\ell=0}^{N} a_{j,\ell} C^{N-\ell} D^\ell,
\]

where \(a_{j,\ell} = \binom{N}{\ell} \sum_{k=1}^{N-1} \lambda_{j,k} \varepsilon_k^\ell\) whenever \(1 \leq \ell \leq N - 1\), \(a_{j,0} = \sum_{k=0}^{N-1} \lambda_{j,k}\) and \(a_{j,N} = \sum_{k=1}^{N} \lambda_{j,k}\).

Our problem admits a solution if the \(\lambda_{j,k}\) can be chosen so that \(a_{j,\ell} = 0\) if \(\ell \neq j\) and \(a_{j,j} = 1\). Let \(\chi_j \in \mathbb{C}^{N+1}\) be the column vector \((\lambda_{j,0}, \ldots, \lambda_{j,N})\), \(e_m = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{C}^{N+1}\) the canonical column vector and define \(\eta_j \in \mathbb{C}^{N+1}\) as \(\eta_j = \binom{N}{j}^{-1} e_{j+1}\), \((j = 0, 1, \ldots, N)\). Finally let \(V\) be the Vandermonde
Suppose that Let Since by Theorem 2.9, Thus Let our problem has been reduced to find a solution of the systems \( A \cdot \chi_j = \eta_j \) for \( j = 0, \cdots, N \). Since \( A \) is regular, we get the solution we were looking for.

Our example above shows that, in particular, for interpolating Blaschke products \( C \) and \( D \)

\[
I(C^2, D^2, (C - D)^2) = J(C^2, D^2, (C - D)^2) = J(C^2, D^2).
\]

It is interesting to compare with the following example.

**Example 1.** — Let \( C, D \) be Blaschke products without common zeros in \( \mathbb{D} \). Suppose that \( I(C, D) \neq H^\infty \). Then for every \( \sigma \in \mathbb{C} \)

\[
I := I(C^2, D^2, (C + \sigma D^2)^2) \neq J(C^2, D^2, (C + \sigma D^2)^2) =: J.
\]

In fact, \( CD \in J \) (because \( |CD| \leq |C|^2 + |D|^2 \)); but if \( CD \in I \), then

\[
CD = xC^2 + yD^2 + t(C^2 + 2\sigma CD + \sigma^2 D^2) = (x + t)C^2 + (y + 2\sigma tC + \sigma^2 tD^2)D^2 = uC^2 + vD^2.
\]

Dividing by \( CD \) yields 1 = \( u\cdot C + v\cdot D \); a contradiction.

**Proposition 6.5.** — Suppose that \( C, D \) and \( B_j \) are functions in \( H^\infty \) such that \( I(B_1, \ldots, B_m) = I(C, D) \). Then \( I(B_1^N, \ldots, B_m^N) \neq J(B_1^N, \ldots, B_m^N) \) whenever \( N \geq m \). Equality is possible if \( N + 1 = m \).

**Proof.** — Since \( B_j \in I(C, D) \) we have that \( |B_j| \leq \kappa(|C| + |D|) \); hence, by Hölder’s inequality, \( |B_j|^N \leq \kappa^N(|C|^N + |D|^N) \). Similarly \( C \in I(B_1, \ldots, B_m) \) implies that \( |C|^N \leq \kappa \sum_{j=1}^m |B_j|^N \). Thus \( J(B_1^N, \ldots, B_m^N) = J(C^N, D^N) \). Therefore, if \( I(B_1^N, \ldots, B_m^N) = J(B_1^N, \ldots, B_m^N) \), we obtain

\[
I(B_1^N, \ldots, B_m^N) = J(C^N, D^N) = I(C^N, C^{-1}D, \ldots, C^{-1}D, D^N).
\]

By Theorem 2.9, \( m \geq N + 1 \). Thus if \( N \geq m \), we have that \( I(B_1^N, \ldots, B_m^N) \) is strictly contained in \( J(B_1^N, \ldots, B_m^N) \). Proposition 6.4 shows that equality is possible for \( m = N + 1 \).

7. Intersections of finitely generated ideals

It is well known that if \( I = I(f_1, \ldots, f_N) \) is a finitely generated ideal, then \( J(f_1, \ldots, f_N) = \bigcap (I + bH^\infty) \), where \( b \) runs through the set of interpolating Blaschke products (see [18]). In [13] it was shown that if \( I = I(b_1^N, b_2^N) \), where \( b_1 \) and \( b_2 \) are interpolating Blaschke products
without common zeros, then there exists interpolating Blaschke products $b_j$ $(j = 3, \ldots, N)$, such that

$$J(b_1^N, b_2^N) = \bigcap_{j=1}^N (I + b_j H^\infty).$$

**Question 2.** Suppose that $I$ is a finitely generated ideal of order $N$. Is it true that $J = \bigcap_{j=1}^N (I + b_j H^\infty)$ for $N$ interpolating Blaschke products $b_j$?

We also point out the following result of Tolokonikov (see [13]).

**Proposition 7.1.** Let $I = \bigcap_{n=1}^N I_n$ be an intersection of finitely generated ideals, $I_n$, each of which contains an interpolating Blaschke product. Then $I$ is finitely generated and $I = J$.

**Proof.** By the theorem of McVoy and Rubel [10], $I$ is finitely generated. Let $J_n$ be the associated $J$-form of $I_n$ and let $f \in J$. Then $f \in J_n$ for each $n$. Since by Theorem 0.1 $I_n = J_n$, we see that $f \in \bigcap J_n = I$. \hfill \square

In the following we will prove some facts for general Blaschke products.

**Proposition 7.2.** Suppose that $B_1, B_2, C_1, C_2$ are Blaschke products without common zeros and let $I = I(B_1B_2, B_1C_2, C_1C_2)$. Then $I$ is finitely generated and $I = J$.

**Proof.** The inclusion $\subseteq$ being trivial, it remains to show $\supseteq$. So let $f \in (I + B_1 H^\infty) \cap (I + C_2 H^\infty)$. Then $f = xC_1C_2 + yB_1 = \tilde{x}B_1B_2 + \tilde{y}C_2$ for some functions $x, y, \tilde{x}, \tilde{y} \in H^\infty$. Hence $B_1(y - \tilde{x}B_2) = C_2(y - xC_1)$. Since $B_1$ and $C_2$ have no common zeros, $\tilde{y} - xC_1 = kB_1$ for some $k \in H^\infty$. Hence $\tilde{y} \in I(C_1, B_1)$ and so $f \in I(B_1B_2, B_1C_2, C_1C_2)$. \hfill \square

**Proposition 7.3.** Let $B_1, B_2, C_1, C_2$ be Blaschke products without common zeros and let

$$I = I(B_1B_2B_3, B_1B_2C_3, B_1C_2C_3, C_1C_2C_3).$$

Suppose that $C_1 \in I(B_1, B_2)$. Then

$$I = (I + B_1 H^\infty) \cap (I + B_2 H^\infty) \cap (I + C_3 H^\infty).$$

**Proof.** As above we have only to show the inclusion $\supseteq$. So let $f \in (I + B_1 H^\infty) \cap (I + B_2 H^\infty) \cap (I + C_3 H^\infty)$. Then

\begin{align*}
(7.1) & \quad f = xB_1 + yC_1C_2C_3 \\
(7.2) & \quad = \tilde{x}C_3 + \tilde{y}B_1B_2B_3 \\
(7.3) & \quad = rB_1C_2C_3 + sC_1C_2C_3 + tB_2.
\end{align*}

From (7.1) and (7.2) we obtain $B_1(x - yB_2B_3) = C_3(\tilde{x} - yC_1C_2)$. Since $B_1$ and $C_3$ have no common zeros, $x - yB_2B_3 = KC_3$. Hence $x \in I(C_3, B_2B_3)$.

Let $C_1 = \alpha B_1 + \alpha' B_2$. Using (7.1) and (7.3), we obtain the following chain of implications:

$$xB_1 + C_1(yC_2C_3 - sC_2C_3) = rB_1C_2C_3 + tB_2 \implies xB_1 + (\alpha B_1 + \alpha' B_2)(yC_2C_3 - sC_2C_3) = rB_1C_2C_3 + tB_2 \implies$$

$$xB_1 + (\alpha B_1 + \alpha' B_2)(yC_2C_3 - sC_2C_3) = rB_1C_2C_3 + tB_2 \implies$$
\[ B_1[x + (\alpha y - \alpha s - r)C_2C_3] = B_2[t - \alpha'C_2C_3(y - s)] \implies x + (\alpha y - \alpha s - r)C_2C_3 = \bar{k}B_2. \]

Hence \( x \in I(B_2, C_2C_3) \). Therefore, by Proposition 7.2

\[ x \in I(B_2, C_2C_3) \cap I(C_3, B_2B_3) = I(B_2B_3, B_2C_3, B_2C_3). \]

From this we deduce that \( f \in I(B_1B_2B_3, B_1B_2C_3, B_1C_2C_3, C_1C_2C_3) \).

We point out that if \( B_1, B_2 \) and \( C_3 \) above are interpolating Blaschke products, then, by Theorem 7.1, we get that \( I = J \). Since the present condition \( C_1 \in I(B_1, B_2) \) is the same as \( C_1 \) being bounded on \( Z_3(B_2) \), we have obtained a second proof for \( N = 3 \) of Proposition 5.2.

**Corollary 7.4.** — Let \( B_1, B_2, C_1, C_2 \) be Blaschke products without common zeros and let

\[ I = I(B_1B_2B_3, B_1B_2C_3, B_1C_2C_3, C_1C_2C_3), \]

respectively,

\[ I^* = I(B_1B_2B_3, B_1B_2C_3, B_1C_2C_3, B_2C_2C_3). \]

Then

1. \( I^* = (I^* + B_1H^\infty) \cap (I^* + B_2H^\infty) \cap (I^* + C_3H^\infty) \).
2. \( I^* = I(B_1, B_2C_2C_3) \cap I(B_2, B_1C_2C_3) \cap I(C_3, B_1B_2B_3) \).
3. \( I = I^* \) if and only if \( I(B_1, B_2) = I(B_1, C_1) \).

If \( B_1, B_2 \) and \( C_3 \) are interpolating Blaschke products, then \( I^* = J^* \), where \( J^* \) is the \( J \)-form of \( I^* \).

**Proof.** — (1), (2) and (3) immediately follow from Proposition 7.3. That \( I^* = J^* \) follows from (1) and Proposition 7.1.

We end this section with the following questions (see also [13]):

**Question 3.** — Let \( I \) be a finitely generated ideal in \( H^\infty \). Does \( I = J \) imply that \( Z(I) \subseteq G \) and that \( \text{ord } I + 1 \) is the minimal number of generators?

**Question 4.** — Is \( I = J \) if and only if \( J \) is a finite intersection of finitely generated order one ideals?

### 8. Addendum

Here we shall prove for the reader’s convenience a structural result on the \( \delta \)-neighborhood of the zero set of a Carleson-Newman Blaschke product that is needed in our proofs above.

Let \( C \) be a collection of closed pseudohyperbolic discs in \( \mathbb{D} \). If \( n \in \mathbb{N} \) and \( D_j \in C \), then we will call \( \{D_1, \ldots, D_n\} \) an \( n \)-chain if \( \cup_{j=1}^n D_j \) is connected. In this case, the number \( n \) is called the length of the chain. If \( \{D_1, \ldots, D_M\} \) is an \( M \)-chain of discs of fixed pseudohyperbolic radius \( \delta \), then \( \bigcup_{j=1}^M D_j \) is contained in a pseudohyperbolic disc of radius \( \rho = \rho(\delta, M) \), where \( \rho \) depends on \( \delta \) and \( M \) only. As \( \rho \) we may take the pseudohyperbolic diameter of the \( M \)-chain \( \{D_\rho(x_j, \delta) : j = 1, \ldots, M\} \) of adjacents disks, where \( x_1 = \delta \) and \( x_{n+1} = f^{[2]}(x_n) \), where
$f^{[n]} := f \circ \cdots \circ f$ is the $n$-th iterate of the function $f(x) = x + \delta \frac{x^2}{1 + x^2}$. We have that $x_j = f^{[2j-1]}(0)$.

Finally, for $M$ fixed, $\rho(\delta, M) \to 0$ as $\delta \to 0$.

If $C$ is an open covering of $[0,1]$ by pseudohyperbolic discs of a fixed radius and such that the centers of the discs do not accumulate inside $\mathbb{D}$, then for every $M \in \mathbb{N}$, there exists a chain of length $M$ in $C$.

The following is a variant of several resuts of this type appearing in literature. The new feature is the concept of $n$-chains introduced above.

**Proposition 8.1.** — Let $B$ be a Carleson-Newman Blaschke product of order $N$ with zero sequence $(z_j)$. Then there exists $\delta_0$ such that for all $0 < \delta \leq \delta_0$ the set $\bigcup_j D_{\rho}(z_j, \delta)$ is a union of pairwise disjoint $n$-chains, for $n \in \{1, \ldots, N\}$, and $B$ is bounded away from zero on $\mathbb{D} \setminus \bigcup_j D_{\rho}(z_j, \delta)$.

**Proof.** — By our hypothesis, $B = B_1 \cdots B_N$, where each $B_j$ is an interpolating Blaschke product. Let $(a_{n,j})_{n \in \mathbb{N}}$ denote the zero sequence of $B_j$. Since each $B_j$ is interpolating, we may choose $\delta > 0$ so small that for each $j$ the discs $\{D(a_{n,j}, \rho(\delta, N+1)) : n \in \mathbb{N}\}$ have disjoint closures. We may assume that $\delta < \rho(\delta, N+1) < \min_{1 \leq j \leq N} \delta(B_j)$.

By Lemma 4.1, $B_j$ has the property that there exists $\eta_j$ such that $|B_j| > \eta_j$ off $\bigcup_n D(a_{n,j}, \delta)$. Therefore $|B| > \eta := \prod_{j=1}^N \eta_j > 0$ outside $\bigcup D_n$, where $D_n$ is the pseudo-hyperbolic disc of radius $\delta$ centered at the zero $z_n$ of $B$. Now, if the family $\{D_n : n \in \mathbb{N}\}$ would contain an $N + 1$-chain, then there would exist $j$ such that the interpolating Blaschke product $B_j$ has at least two zeros in a disc of radius $\rho(\delta, N + 1)$. This contradicts the assumption that the uniform separation constant of $B_j$ is bigger than $\rho(\delta, N + 1)$. Hence $\{D_n : n \in \mathbb{N}\}$ is a family of pairwise disjoint chains of length at most $N$.

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**References**


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