COMMON BOUNDED UNIVERSAL FUNCTIONS FOR COMPOSITION OPERATORS

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Abstract. Let $\mathcal{A}$ be the set of automorphisms of the unit disk with 1 as attractive fixed point. We prove that there exists a single Blaschke product that is universal for every composition operator $C_\phi$, $\phi \in \mathcal{A}$, acting on the unit ball of $H^\infty(\mathbb{D})$.

1. Introduction

This paper is devoted to the construction of common universal functions for some uncountable families of composition operators on the unit ball $\mathcal{B}$ of $H^\infty(\mathbb{D})$. If $\phi : \mathbb{D} \to \mathbb{D}$ is an analytic self-map of the unit disk $\mathbb{D}$, the composition operator $C_\phi : f \mapsto f \circ \phi$ acts continuously on $\mathcal{B}$ (note that $\mathcal{B}$ will always be endowed with the topology of uniform convergence on compact sets). A function $f \in \mathcal{B}$ is said to be $\mathcal{B}$-universal for $C_\phi$, (or just universal, if no ambiguities arise) if $\mathcal{O}(f) = \{f \circ \phi^n : n \geq 0\}$ is dense in $\mathcal{B}$, where $\phi^n = \phi \circ \phi \circ \ldots \circ \phi$ denotes the $n$-th iterate of $\phi$. The operator $C_\phi$ is $\mathcal{B}$-universal if it admits a $\mathcal{B}$-universal function, and this happens ([3]) if and only if $\phi$ is a hyperbolic or parabolic automorphism of the unit disk. In this case the universal function can be chosen to be a Blaschke product. Our aim in this paper is to construct common universal Blaschke products for some uncountable families of composition operators $C_\phi$ acting on $\mathcal{B}$, the $\phi$’s being hyperbolic and parabolic automorphisms of $\mathbb{D}$.

Results on universal Blaschke products first appear in a paper by Heins [10]. A general theory of universal Blaschke products and their behaviour on the maximal ideal space of $H^\infty$ was developed in [8] and [11]. Finally, these functions were the building blocks for studying $\mathcal{B}$-universality for sequences of composition operators $(C_{\phi_n})$ in [3].

Our study of universal Blaschke products in the present paper is motivated by previous results of common hypercyclicity of [1], [4] and [5]. Indeed the operators $C_\phi$ act boundedly on different spaces, such as the space $\mathcal{H}(\mathbb{D})$ of holomorphic functions on $\mathbb{D}$, or the Hardy spaces $H^p(\mathbb{D})$, $1 \leq p < +\infty$, and
when $\phi$ is a hyperbolic or parabolic automorphism, $C_\phi$ is hypercyclic on $\mathcal{H}(D)$ (resp. $H^p(D)$), i.e. there exists a function $f \in \mathcal{H}(D)$ (resp. $f \in H^p(D)$) such that $\mathcal{O}(f)$ is dense in $\mathcal{H}(D)$ (resp. $H^p(D)$). It is then natural to ask about the existence of a function $f$ which would be hypercyclic for all composition operators $C_\phi$. Since each function in $H^p(D)$ has a radial limit almost everywhere on the unit circle $T$, such a common hypercyclic function cannot exist on $H^p(D)$: if $\mathcal{A}$ is a family of hyperbolic or parabolic automorphisms of $D$, the fact that the family $(C_\phi)_{\phi \in \mathcal{A}}$ has a common hypercyclic vector necessarily implies that the subset $B$ of $T$ consisting of all the attractive fixed points of the automorphisms $\phi \in \mathcal{A}$ has Lebesgue measure zero. Hence a natural family to consider is $(C_\phi)_{\phi \in \mathcal{A}_0}$, where $\mathcal{A}_0$ is the class of hyperbolic or parabolic automorphisms of $D$ with $1$ as attractive fixed point. Then this restricted family of composition operators acting on $H^p(D)$ admits a common hypercyclic vector ([4] or [5]).

We deal here with the same question, but our underlying space is now the unit ball $B$ of $H^\infty(D)$. The main difficulty in this new setting lies in the fact that all the techniques of [1], [4] or [5] are “additive” and strongly use the linearity of the space, making it difficult to control the $H^\infty$-norm of the functions which are constructed. We have to use “multiplicative” techniques instead to prove the following theorem, which is the main result of this paper:

**Theorem 1.** There exists a Blaschke product $B$ which is universal for all composition operators $C_\phi$ associated to hyperbolic or parabolic automorphisms of $D$ with $1$ as attractive fixed point.

The proof of this result uses an argument of Costakis and Sambarino ([6]).

The hyperbolic and parabolic cases will be treated separately (in Sections 2 and 3 respectively), the hyperbolic case being as usual easier than the parabolic one, since we have a better control of the rate of convergence of the iterates to the attractive fixed point.

### 2. The hyperbolic case

We first consider for $\lambda > 1$ the family of hyperbolic automorphisms

$$z \mapsto \frac{z + \frac{1}{\lambda+1}}{1 + z \frac{1}{\lambda+1}}$$

of $D$ with $1$ as attractive fixed point and $-1$ as repulsive fixed point. The action of such an automorphism is best understood when considered on the right half-plane $C_+ = \{w \in \mathbb{C} : \Re w > 0\}$: if $\sigma : D \to C_+$ is the Cayley map defined by $\sigma(z) = \frac{1+z}{1-z}$, such an automorphism is conjugated via $\sigma$ to a dilation $\varphi_\lambda : w \mapsto \lambda w$, where $\lambda > 1$. We will denote by $\phi_\lambda$ the hyperbolic automorphism of $D$ such that $\phi_\lambda = \sigma^{-1} \circ \varphi_\lambda \circ \sigma$. A general hyperbolic automorphism with $1$ as attractive fixed point has the form $\phi_{\lambda,\beta} = \sigma^{-1} \circ \varphi_{\lambda,\beta} \circ \sigma$,
where \( \varphi_{\lambda, \beta} \) acts on \( \mathbb{C}_+ \) as \( \varphi_{\lambda, \beta}(w) = \lambda(w - i\beta) + i\beta, \lambda > 1, \beta \in \mathbb{R} \). We first show that the parameters \( \beta \) play essentially no role in this problem.

**Lemma 2.** Let \( B \) be a Blaschke product which is universal for \( C_{\phi_{\lambda}}, \lambda > 1 \). For any \( \beta \in \mathbb{R}, B \) is universal for \( C_{\phi_{\lambda, \beta}} \).

**Proof.** Let \( f \in B \) and let \( K \) be a compact subset of \( \mathbb{D} \). For \( z \in K \), we define

\[
\begin{align*}
  z_1(n) &= \sigma^{-1}(\lambda^n(\sigma(z) - i\beta) + i\beta) = \phi_{\lambda, \beta}^{[n]}(z) \\
  z_2(n) &= \sigma^{-1}(\lambda^n(\sigma(z) - i\beta)) = \phi_{\lambda}^{[n]}(\sigma^{-1}(\sigma(z) - i\beta)).
\end{align*}
\]

It is easy to show that there exists a constant \( C_1 \) which depends only on \( K \) and \( \beta \) such that

\[
|z_1(n) - z_2(n)| \leq \frac{C_1}{\lambda^{2n}}.
\]

In fact, if \( w_1(n) = \lambda^n(\sigma(z) - i\beta) + i\beta \) and \( w_2(n) = \lambda^n(\sigma(z) - i\beta) \), then

\[
|z_1(n) - z_2(n)| = \left| \int_{[w_1(n), w_2(n)]} \frac{2}{(1 + |w|)^2} dw \right| \leq |\beta| \max_{w \in [w_1(n), w_2(n)]} \frac{2}{1 + |w|^2} \leq \frac{2}{\lambda^{2n}[\min\{\Re \sigma(z) : z \in K\}]^2} \leq \frac{C_1}{\lambda^{2n}}.
\]

On the other hand, there is another constant \( C_2 \), depending only on \( K \) and \( \beta \) such that

\[
|z_1(n)| \leq 1 - \frac{C_2}{\lambda^n} \quad \text{and} \quad |z_2(n)| \leq 1 - \frac{C_2}{\lambda^n}.
\]

This can be seen in the following way:

\[
|1 - z_j(n)| = 1 - \frac{|w_j(n) - 1|}{|w_j(n) + 1|} = 2/|w_j(n) + 1| \geq \frac{C_2}{\lambda^n}.
\]

Since \( B \) belongs to \( H^\infty(\mathbb{D}) \), Cauchy’s inequalities show that \( B(z_1(n)) - B(z_2(n)) \) converges uniformly on \( K \) to 0. In fact

\[
|B(z_1(n)) - B(z_2(n))| \leq \int_{[z_1(n), z_2(n)]} |B'(\xi)|(|1 - |\xi|^2|) d\xi
\]

\[
\leq \frac{1}{\min\{1 - |z_1(n)|^2, 1 - |z_2(n)|^2\}} \leq \frac{C_3}{\lambda^n}.
\]

On the other hand, since \( B \) is universal, there exists a sequence \( (n_k) \) such that \( B \circ \phi_{\lambda}^{[n_k]}(\sigma^{-1}(\sigma(z) - i\beta)) \) converges uniformly to \( f \) on \( K \) (the map \( z \mapsto \sigma^{-1}(\sigma(z) - i\beta) \) is an automorphism of \( \mathbb{D} \)). We conclude that \( B \circ \phi_{\lambda, \beta}^{[n_k]} \) converges uniformly on \( K \) to \( f \). \[\Box\]
Proof of Lemma 3. Let \( f \) be a finite Blaschke product such that \( f(1) = f(-1) = 1 \).

For every compact subset \( K \) of \( \mathbb{D} \) and each interval \( [a, b] \subseteq [1, \infty] \), there exists a positive constant \( M \) depending on \( K \), \( f \) and \( a \) such that for every \( j = 1, \ldots, q \) and every \( \lambda \in \{\lambda_j, \lambda_{j+1}\} \) the following assertions are true:

1. For every \( l < j \),
   \[
   \left\| C_{\phi_{\lambda_j}}^{(jN)} C_{\phi_{\lambda_j}}^{(-1N)} (f) - 1 \right\|_K \leq M a^{-(l-1)N};
   \]
2. For every \( l > j \),
   \[
   \left\| C_{\phi_{\lambda_j}}^{(jN)} C_{\phi_{\lambda_j}}^{(-1N)} (f) - 1 \right\|_K \leq M a^{-(l-j)N};
   \]
3. \[
   \left\| C_{\phi_{\lambda_j}}^{(jN)} C_{\phi_{\lambda_j}}^{(-1N)} (f) - f \right\|_K \leq M \delta.
   \]

We will use repeatedly the following fact, which follows from the Schwarz-Pick estimates:

Lemma 4. Let \( u \in \mathcal{B} \). Then for every \( z \in \mathbb{D} \),

\[
|u(z) - 1| \leq \frac{1+|z|}{1-|z|} |u(0) - 1|.
\]

Proof. We obviously have that \( \frac{u(z)-u(0)}{1-u(0)u(z)} = zg(z) \) for some \( g \in \mathcal{B} \). Hence

\[
|u(z) - 1| \leq |u(z) - u(0)| + |u(0) - 1| \leq |z| |1-\overline{u(0)}u(z)| + |u(0) - 1|
\leq |z||(1-u(0)) + u(0)(1-u(z))| + |u(0) - 1|.
\]

Therefore \( |u(z) - 1| \leq |1-|z|| \leq |1-u(0)|(1+|z|) \) for every \( z \in \mathbb{D} \).

Thus in order to prove assertions 1 and 2 above, for instance, it suffices to control in a suitable way the quantities \( f(\phi_{\lambda_{1l}}^{([N])}(\phi_{\lambda_{1l}}^{[N]}(0))) \).

Proof of Lemma 3. For every \( z \in \mathbb{D} \) we have

\[
C_{\phi_{\lambda_j}}^{(jN)} C_{\phi_{\lambda_j}}^{(-1N)} (f)(z) = f \left( \sigma^{-1} \left( \frac{\lambda_j N}{\lambda_j} \sigma(z) \right) \right).
\]

Since \( f(1) = 1 \) and \( f \) is Lipschitz with constant \( C \) up to the boundary of \( \mathbb{D} \), we have

\[
\left| f(\phi_{\lambda_{1l}}^{([N])}(\phi_{\lambda_{1l}}^{[N]}(0))) - 1 \right| \leq C \left| \sigma^{-1} \left( \frac{\lambda_j N}{\lambda_j} \right) - 1 \right| = \frac{2C}{\lambda_j N + 1}.
\]
Assertion 1 follows from this estimate: since $l < j$,
\[ \frac{\lambda^j N}{\lambda_l^N} \geq \frac{\lambda^j N}{\lambda_j^{N-1}} \geq \lambda_{j-1}^{(j-1)N} \left( 1 + \frac{\delta}{\lambda_{j-1}N} \right)^N j \geq \lambda_{j-1}^{(j-1)N} \geq a^{(j-1)N}. \]

By Lemma 4, there exists a positive constant $M_1$ such that
\[ \|C_{\phi_l}^{jN} C_{\phi_{-j}}^{-jN} (f) - 1\|_{K} \leq \frac{M_1}{a^{(j-l)N}} \text{ for } l < j. \]

Assertion 2 is proved in the same fashion, using this time the fact that $f(-1) = 1$, so that
\[ \left| f(\phi_{-j}^{[N]} \circ \phi_j^{[N]}(0)) - 1 \right| \leq 2C \frac{\lambda^j N}{\lambda_l^N + 1} \]
and that for $l > j$,
\[ \frac{\lambda^j N}{\lambda_l^N} \leq \lambda_{j+1}^{(j-1)N} \leq a^{(j-l)N}. \]

As to assertion 3, we have for every $z \in \mathbb{D}$
\[ \left| C_{\phi_l}^{jN} C_{\phi_{-j}}^{-jN} (f)(z) - f(z) \right| \leq C |\sigma^{-1} \left( \frac{\lambda^j N}{\lambda_l^N} \sigma(z) \right) - z| \leq C \frac{|\lambda^j N|}{|\lambda_l^N| - 1} \cdot \frac{2|\sigma(z)|}{|\lambda^j N \sigma(z) + 1|^2}. \]

Since $\frac{|\lambda^j N \sigma(z) + 1|}{|\lambda^j N \sigma(z) + 1|}$ is bigger than its real part, which is bigger than 1, and since
\[ 0 \leq (\lambda_j^N)^{-1} - 1 \leq (1 + \frac{\delta}{\alpha N})^N - 1 \leq 2\delta/a \text{ when } \delta \text{ is small enough}, \]
we have
\[ ||C_{\phi_l}^{jN} C_{\phi_{-j}}^{-jN} (f) - f||_{K} \leq M_3 \delta \]
for some positive constant $M_3$. □

We need a last lemma.

**Lemma 5.** The finite Blaschke products $f$ such that $f(1) = f(-1) = 1$ are dense in $\mathcal{B}$ (for the topology of uniform convergence on compact sets).

**Proof.** We use Carathéodory’s theorem that the set of finite Blaschke products is dense in $\mathcal{B}$ and a special case of an interpolation result given in [9, p. Lemma 2.10] that tells us that for every $\varepsilon > 0$, every compact subset $K \subseteq \mathbb{D}$ and $\alpha, \beta \in \mathbb{T}$ there exists a finite Blaschke product $B_1$ satisfying $B_1(1) = \alpha, B_1(-1) = \beta$ and $||B_1 - 1||_{K} < \varepsilon$. Thus, given $f \in \mathcal{B}$ and a finite Blaschke product $B_0$ that is close to $f$ on $K$, we solve the interpolation problem with $\alpha = B_0(1)$.
and \( \beta = B_0(-1) \) and set \( B = B_0B_1 \), in order to get the desired Blaschke product. \( \square \)

With these two lemmas in hand, we prove the following proposition:

**Proposition 6.** Let \((f_k)_{k \geq 1}\) be a dense sequence of finite Blaschke products with \( f_k(1) = f_k(-1) = 1 \). Let \((K_k)_k\) be an exhaustive sequence of compact subsets of \( \mathbb{D} \), and \(([a_k, b_k])_{k \geq 1}\) an increasing sequence of compact intervals such that
\[
\bigcup_{k \geq 1} [a_k, b_k] = ]1, +\infty[.
\]

There exist
\begin{itemize}
  \item a sequence \((B_n)_{n \geq 1}\) of finite Blaschke products;
  \item an increasing sequence \((p_n)_{n \geq 1}\) of positive integers
\end{itemize}
such that the following are satisfied for every \( k \geq 1 \):
\begin{enumerate}
  \item \( B_k(1) = 1 \);
  \item \(||B_k - 1||_{K_k} < 2^{-k} \);
  \item for every \( \lambda \in [a_k, b_k] \), there exists an integer \( n(\lambda) \leq p_k \) such that for every \( i \geq k \),
\end{enumerate}

\[
|\left| C_n(\lambda)(B_1 \ldots B_i) - f_k \right|_{K_k} | < 2^{-k}.
\]

As a corollary, we obtain:

**Corollary 7.** There exists a Blaschke product \( B \) which is universal for all the composition operators \( C_{\phi, \lambda} \), \( \lambda > 1 \), \( \beta \in \mathbb{R} \).

**Proof.** Consider \( B = \prod_{n=1}^{\infty} B_n \); this is a convergent Blaschke product by property 2, and going to the limit as \( i \) goes to infinity in equation (1) implies that for every \( \lambda \in [a, b] \) and \( k \) large enough \((\{a, b\} \subseteq [a_k, b_k])\),
\[
|\left| C_{\phi, \lambda}^n(B) - f_k \right|_{K_k} | \leq 2^{-k}.
\]

Since the family \((f_k)_{k \geq 1}\) is locally uniformly dense in \( \mathbb{D} \), this proves the universality of \( B \) for \( C_{\phi, \lambda} \), hence for \( C_{\phi, \lambda, \beta} \). \( \square \)

We turn now to the proof of Proposition 6:

**Proof.** The proof is done by induction on \( k \). We consider a first partition \( a_1 = \lambda_1 < \lambda_2 < \ldots < \lambda_{q_1} = b_1 \) of \([a_1, b_1]\) with parameters \( N_1 \) and \( \delta_1 \), and the finite Blaschke product
\[
B_1 = \prod_{l=1}^{q_1} C_{\phi, \lambda_i}^{-N_1}(f_1).
\]

We have \( B_1(1) = 1 \). Since \( f_1(-1) = 1 \), \( C_{\phi, \lambda_1}^{-N_1}(f_1) \) tends to 1 uniformly on compact sets as \( N_1 \) tends to infinity, and if \( N_1 \) is large enough,
\[
||B_1 - 1||_{K_1} < 2^{-1}.
\]
Since $|\prod_{j=1}^{l} a_j - \prod_{j=1}^{l} b_j| \leq \sum_{j=1}^{l} |a_j - b_j|$ whenever $a_j, b_j \in \mathbb{D}$, for every $j = 1, \ldots, q_1$ and every $\lambda \in [\lambda_j, \lambda_{j+1}]$, we have for any compact subset $K$ of $\mathbb{D}$

$$\left\| C_{\phi_\lambda}^{N_k}(B_1) - f_1 \right\|_K \leq \sum_{l=1,l\neq j}^{q_1} \left\| C_{\phi_\lambda}^{N_k}C_{\phi_{\lambda_l}}^{lN_k}(f_1) - 1 \right\|_K + \left\| C_{\phi_\lambda}^{N_k}C_{\phi_{\lambda_l}}^{lN_k}(f_1) - f_1 \right\|_K.$$

But by Lemma 3, the quantity on the righthand side is less than

$$\sum_{l=1,l\neq j}^{q_1} \frac{M}{a_{l}^{l-1N_k}} + M \delta_1 \leq 2M \sum_{k=1}^{\infty} \frac{1}{a_{k}^{kN_k}} + M \delta_1.$$

Thus if $N_k$ is large enough and $\delta_1$ small enough

$$\left\| C_{\phi_\lambda}^{N_k}(B_1) - f_1 \right\|_{K_k} < 2^{1}.$$

We now fix $N_k$ large enough and $\delta_1$ small enough so that inequalities (2) and (3) are satisfied. It is easy to check that assertions 2 and 3 of Proposition 6 are satisfied with $p_1 = q_1N_k$ and $n_1(\lambda) = jN_k$ for $\lambda \in [\lambda_j, \lambda_{j+1}]$. This terminates the first step of the construction.

If now the construction has been carried out until step $k - 1$, we consider again a partition $a_k = \lambda_1 < \ldots < \lambda_{q_k} = b_k$ of $[a_k, b_k]$ with parameters $\delta_k$ and $N_k$, and set

$$B_k = \prod_{l=1}^{q_k} C_{\phi_{\lambda_k}}^{lN_k}(f_k),$$

so that $B_k$ is a finite Blaschke product with $B_k(1) = 1$. Just as above if $N_k$ is large enough and $\delta_k$ small enough, we have for every $j \leq q_k$, every $\lambda \in [\lambda_j, \lambda_{j+1}]$

$$\left\| C_{\phi_\lambda}^{N_k}(B_k) - f_k \right\|_{K_k} < 2^{-(k+1)}$$

and

$$\left\| B_k - 1 \right\|_{K_k} < 2^{-k}.$$

Because $B_1(1) = \cdots = B_{k-1}(1) = 1$, we can also choose simultaneously $N_k$ large enough so that $C_{\phi_\lambda}^{N_k}(B_1 \cdots B_{k-1})$ is very close to 1 on $K_k$. This gives (1) for $i = k$.

It remains to check that if $r \leq k - 1$, $\lambda \in [a_r, b_r]$,

$$\left\| C_{\phi_\lambda}^{rN_k}(B_1 \cdots B_{k-1}B_k) - f_r \right\|_{K_r} < 2^{-r}.$$

We already know that

$$\left\| C_{\phi_\lambda}^{rN_k}(B_1 \cdots B_{k-1}) - f_r \right\|_{K_r} < 2^{-r},$$
and since $B_k$ can be made arbitrarily close to 1 on any compact set if $N_k$ is large enough, we also choose $N_k$ so that $\|B_k - 1\|_K$ is small enough, where

$$K = \bigcup_{r \leq k-1, \lambda \in [a_r, b_r]} \phi_\lambda^{n_r}(K_r),$$

and then assertions 2 and 3 are satisfied at step $k$. \hfill \Box

### 3. The parabolic case

We consider now the family of parabolic automorphisms of $\mathbb{D}$ with 1 as attractive fixed point. If $T_\lambda : \mathbb{C}_+ \to \mathbb{C}_+$ is the translation defined by $w \mapsto w + i\lambda, \lambda \in \mathbb{R} \setminus \{0\}$, then such parabolic automorphisms have the form $\psi_\lambda(z) = \sigma^{-1} \circ T_\lambda \circ \sigma$. Our aim in this section is to construct a Blaschke product which is universal for all composition operators $(C_{\psi_\lambda}), \lambda > 0$. This is more difficult than the hyperbolic case, because we have no suitable analog of Lemma 3: the estimate we get has the form

$$\left| C_{jN}^{\psi_\lambda} - C_{lN}^{\psi_\lambda} (f) - 1 \right|_K \leq M \frac{|j - l|}{N},$$

and the series on the righthand side is not convergent when we sum over all $l \neq j$.

In other words if $K$ is any compact set, the sets $\psi_\lambda^{[n]}(K)$ go towards the point 1 at a rate of $1/n$, which is too slow. This difficulty was tackled for the study of common hypercyclicity on the Hardy space $H^2(\mathbb{D})$ by using either a fine analysis of properties of disjointness in [4] or probabilistic ideas in [5]. Here we use in a crucial way the tangential convergence of the sequence $\psi_\lambda^{[n]}(0)$ to the boundary. Indeed, the series $\sum_n (1 - |\psi_n(0)|)$ is summable, whereas the series $\sum_n |1 - \psi_n(0)|$ is not. The following lemma will play the same role as Lemma 3 in the hyperbolic case. We keep the notation of Section 2 and use the same kind of decomposition $a = \lambda_1, \ldots, \lambda_q = b$ of a compact sub-interval $[a, b]$ of $]0, +\infty[$.

**Lemma 8.** Let $f$ be a finite Blaschke product such that $f(1) = 1$. For every compact subset $K$ of $\mathbb{D}$, there exists a positive constant $M$ depending on $f$ and a such that for every $j = 1, \ldots, q$ and every $\lambda \in [\lambda_j, \lambda_{j+1}]$ the following assertions are true:

1. For every $l < j$, $\left| C_{jN}^{\psi_\lambda} - C_{lN}^{\psi_\lambda} (f) - 1 \right|_K \leq \frac{M}{(j-l)N^2}$;
2. For every $l > j$, $\left| C_{jN}^{\psi_\lambda} - C_{lN}^{\psi_\lambda} (f) - 1 \right|_K \leq \frac{M}{(l-j)N^2}$;
3. $\left| C_{jN}^{\psi_\lambda} - C_{jN}^{\psi_\lambda} (f) - f \right|_K \leq M \delta$.

**Proof.** In order to prove assertions 1 and 2, it suffices to work at the point 0. Since the modulus of $f$ is equal to 1 on $\mathbb{T}$, and since the operators commute, we have
\begin{equation}
\left| f(\psi_{\lambda}^{[jN]} \circ \psi_{\lambda}^{[lN]}(0)) \right| - 1 = \left| f(\psi_{\lambda}^{[jN]} \circ \psi_{\lambda}^{-[lN]}(0)) \right| - \left| f\left(\frac{\psi_{\lambda}^{[jN]} \circ \psi_{\lambda}^{-[lN]}(0)}{\psi_{\lambda}^{[jN]} \circ \psi_{\lambda}^{-[lN]}(0)}\right)\right|.
\end{equation}

Since $f$ is $C$-Lipschitz on $D$ for some positive constant $C$, this quantity is less than
\begin{equation}
C \left| \psi_{\lambda}^{[jN]} \circ \psi_{\lambda}^{-[lN]}(0) - \psi_{\lambda}^{[jN]} \circ \psi_{\lambda}^{-[lN]}(0) \right| = C \left( 1 - \left| \psi_{\lambda}^{[jN]} \circ \psi_{\lambda}^{-[lN]}(0) \right| \right).
\end{equation}

An easy computation shows that
\[ \psi_{\lambda}^{[jN]} \circ \psi_{\lambda}^{-[lN]}(0) = iN(j\lambda - l\lambda) + iN(j\lambda - l\lambda), \]
Observe that $|j\lambda - l\lambda| \geq |j - l|$. This gives
\[ 1 - \left| \psi_{\lambda}^{[jN]} \circ \psi_{\lambda}^{-[lN]}(0) \right| \leq 1 - \frac{(j\lambda - l\lambda)^2N^2}{4 + (j\lambda - l\lambda)^2N^2} \leq \frac{C_1}{N^2(j - l)^2} \]
for some positive constant $C_1$ which does not depend on $\lambda$. Now equation (4) implies that for $l \neq j$,
\[ \left| \left| f(\psi_{\lambda}^{[jN]} \circ \psi_{\lambda}^{-[lN]}(0)) \right| - 1 \right| \leq \frac{C_2}{N^2(j - l)^2}. \]

This proves assertions 1 and 2 of Lemma 8. Assertion 3 is proved in the same way as in Lemma 3, writing for $\lambda \in [\lambda_j, \lambda_{j+1}]$
\[ \left| C_{\psi_{\lambda}}^{jN} f(z) - f(z) \right| \leq CjN|\lambda_j - \lambda| \leq CjN\frac{\delta}{(j + 1)N} \leq C'\delta. \]

The following proposition is the main ingredient of the proof:

\textbf{Proposition 9.} Let $f$ be a finite Blaschke product, $[a, b] \subset [0, +\infty]$, $m_0$ a positive integer, $K$ a compact subset of $D$ and $\varepsilon > 0$. There exist an integer $m$, integers $(n(\lambda))_{\lambda \in [a, b]}$ with $n(\lambda) \in [m_0, m]$ and a finite Blaschke product $B$ such that
\begin{enumerate}
  \item $B(1) = 1$;
  \item $\|B - 1\|_K < \varepsilon$;
\end{enumerate}
\( C_{\psi_{\lambda}}(B) = u_{\lambda}v_{\lambda} \) where \( u_{\lambda} \) and \( v_{\lambda} \) belong to \( B \), \( \| u_{\lambda} - f \|_K < \varepsilon \) and \( |v_{\lambda}(0)| > 1 - \varepsilon \).

**Proof.** We use again the decomposition \( \lambda_1 = a, \lambda_2 = a + \frac{\delta}{2N}, \ldots, \lambda_q = b \), where \( \delta > 0 \) and \( N \geq m_0 \), will be fixed during the proof. Consider the Blaschke product

\[
B_1 = \prod_{l=1}^q C_{\psi_{\lambda_l}}^{-1}(f).
\]

For \( \lambda \in [\lambda_j, \lambda_{j+1}] \), we have

\[
C_{\psi_{\lambda}}^{(j)}(B) = C_{\psi_{\lambda}}^{(j)}(f) \left( \prod_{l \neq j} C_{\psi_{\lambda_l}}^{(j)}(f) \right) = u_{1,\lambda}v_{1,\lambda}
\]

with \( u_{1,\lambda} = C_{\psi_{\lambda}}^{(j)}(f) \) and \( v_{1,\lambda} = \prod_{l \neq j} C_{\psi_{\lambda_l}}^{(j)}(f) \). By assertion 3 of Lemma 8, \( \| u_{1,\lambda} - f \|_K \leq M\delta < \varepsilon \) if \( \delta \) is small enough. Moreover, still by Lemma 8,

\[
1 - |v_{1,\lambda}(0)| = 1 - \prod_{l \neq j} |f(\psi_{\lambda}^{(j)} \circ \psi_{\lambda_l}^{-1}(0))|
\]

\[
\leq \sum_{l \neq j} \left( 1 - |f(\psi_{\lambda}^{(j)} \circ \psi_{\lambda_l}^{-1}(0))| \right)
\]

\[
\leq C' \frac{N}{N^2}
\]

for some positive constant \( C' \). Thus if \( N \) is large enough, \( |v_{1,\lambda}(0)| > 1 - \varepsilon \).

To conclude, it remains to observe that the same proof leads to

\[
|B_1(0)| \geq 1 - \frac{C''}{N^2}
\]

for some positive constant \( C'' \), so that using Lemma 4 and adjusting \( N \) large enough, there exists a real number \( \theta \) such that \( \| e^{i\theta}B_1 - 1 \|_K < \varepsilon \). If we set \( B_2 = e^{i\theta}B_1 \), then \( B_2 \) satisfies the conclusions of the proposition (setting \( u_{2,\lambda} = u_{1,\lambda} \) and \( v_{2,\lambda} = e^{i\theta}v_{1,\lambda} \), except that we are not sure that \( B_2(1) = 1 \).

To conclude, let \( F \) be a finite Blaschke product such that \( F \) is very close to 1 on a big compact set \( L \subset \mathbb{D} \) and \( F(1) = B_2(1) \). Then \( B = FB_2 \) is the finite Blaschke product we are looking for. Indeed, setting \( u_\lambda = u_{2,\lambda} \) and \( v_\lambda = C_{\psi_{\lambda}}^{(\lambda)}(F)v_{2,\lambda} \), condition 3. is satisfied, provided \( L \) is big enough to contain each \( \psi_{\lambda}^{(\lambda)}(0) \).

\( \square \)

We can now proceed with the construction:
Proposition 10. Let \((f_k)_{k \geq 1}\) be a dense sequence of finite Blaschke products with \(f_k(1) = 1\). Let \((K_k)\) be an exhaustive sequence of compact subsets of \(D\), and \([a_k, b_k])_{k \geq 1}\) an increasing sequence of compact intervals such that
\[
\bigcup_{k \geq 1} [a_k, b_k] = [1, +\infty[.
\]
There exist finite Blaschke products \((B_k)\), integers \((m_k)\), and other integers \((n_k(\lambda))_{\lambda \in [a_k, b_k]}\) with \(n_k(\lambda) \leq m_k\) and such that
\[
\begin{align*}
(1) & \quad B_k(1) = 1; \\
(2) & \quad \|B_k - 1\|_{K_k} < 2^{-k}; \\
(3) & \quad \text{for every } j < k, \text{ every } \lambda \in [a_k, b_k], |B_j \circ \psi_{\lambda}^{[n_k(\lambda)]}(0) - 1| < 2^{-k}; \\
(4) & \quad \text{for every } j < k, \text{ every } \lambda \in [a_j, b_j], |B_k \circ \psi_{\lambda}^{[n_j(\lambda)]}(0) - 1| < 2^{-k}; \\
(5) & \quad \text{for every } \lambda \in [a_k, b_k], C_{\psi_{\lambda}}^{n_k(\lambda)}(B_k) = u_{k, \lambda} v_{k, \lambda} \text{ where}
\end{align*}
\]
\[
\|u_{k, \lambda} - f_k\|_{K_k} < 2^{-k} \text{ and } |v_{k, \lambda}(0)| > 1 - 2^{-k}.
\]
Proof. The first step of the construction follows directly from Proposition 9. Now we assume that the construction has been done until step \(k - 1\) and show how to complete step \(k\). By continuity at the point 1 of the functions \((B_j)_{j < k}\), we choose an integer \(m\) such that, for every \(\lambda \in [a_k, b_k]\), for any \(n \geq m\), \(|B_j \circ \psi_{\lambda}^{[n]}(0) - 1| < 2^{-k}\). We then set
\[
K = K_k \cup \bigcup_{j < k, \lambda \in [a_j, b_j], n \leq m_j} \{\psi_{\lambda}^{[n]}(0)\}.
\]
The function \(B_k\) is then given immediately by Proposition 9.

Corollary 11. There exists a Blaschke product \(B\) which is universal for all the composition operators \(C_{\psi_{\lambda}}\), \(\lambda > 0\).

Proof. Set
\[
B = \prod_{l \geq 1} B_l,
\]
which is a convergent Blaschke product by assertion 2 of Proposition 10. We claim that \(B\) is \(B\)-universal with respect to every composition operator \(C_{\psi_{\lambda}}\). Indeed, fix \(\lambda > 0\) and \(k_0\) such that \(\lambda \in [a_{k_0}, b_{k_0}]\). Let \(g\) be a universal function for this particular operator \(C_{\psi_{\lambda}}\). Using the notation of Proposition 10, let \((p_k)\) be an increasing sequence of integers such that \(f_{p_k}\) converges uniformly to \(g\) on compact subsets of \(D\). Now we decompose
\[
C_{\psi_{\lambda}}^{n_{p_k}}(B) = C_{\psi_{\lambda}}^{n_{p_k}}(B_{p_k}) \left( \prod_{j \neq p_k} B_j \circ \psi_{\lambda}^{[n_{p_k}(\lambda)]} \right) := u_{p_k, \lambda} v_{p_k, \lambda} w_{p_k, \lambda}
\]
where \(C_{\psi_{\lambda}}^{[n_{p_k}(\lambda)]}(B_{p_k}) = u_{p_k, \lambda} v_{p_k, \lambda}\) is the decomposition of Proposition 9. From assertions 3 and 4 of Proposition 10, we get that \(w_{p_k, \lambda}(0)\) tends to 1 (see [2]...
for details), so that (cf. Fact 4) \( w_{p_k, \lambda} \) converges uniformly on compacta to 1. Taking a subsequence if necessary, we can assume that \( v_{p_k, \lambda}(0) \) converges to some unimodular number \( e^{i\theta} \), and by Fact 4 again we have uniform convergence on compacta. Thus \( C_{\psi_{\lambda}}(B) \) converges uniformly to the function \( e^{i\theta}g \) on compacta. Since the function \( e^{i\theta}g \) is universal for \( C_{\psi_{\lambda}} \), this implies that \( B \) is universal for \( C_{\psi_{\lambda}} \) too, and this terminates the proof of Corollary 11.

The proof of Theorem 1 is now concluded by “intertwining” the two proofs of the hyperbolic and parabolic cases: the common universal Blaschke product has the form

\[
B = \prod_{l \geq 1} B_l
\]

where the \( B_l \)'s are finite Blaschke products satisfying a number of properties: \( B_1 \) is constructed using Proposition 6, then \( B_2 \) using Proposition 10, then \( B_3 \) using Proposition 6 again, etc... taking care at each step not to destroy what has been done previously. Details are left to the reader.

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